ADVANCES IN APPLIED MECHANICS

EDITED BY

RICHARD VON MISES
Harvard University
Cambridge, Massachusetts

THEODORE VON KÁRMÁN
Columbia University
New York, New York

VOLUME II

1951

ACADEMIC PRESS INC., PUBLISHERS
NEW YORK, N.Y.

First Printing, 1951
Second Printing, 1962

ACADEMIC PRESS INC.
111 FIFTH AVENUE
NEW YORK 3, N. Y.

United Kingdom Edition
Published by
ACADEMIC PRESS INC. (LONDON) LTD.
17 OLD QUEEN STREET, LONDON SW 1

PRINTED IN THE UNITED STATES OF AMERICA

Contributors to Volume II

R. A. Clark, *Massachusetts Institute of Technology, Cambridge, Massachusetts*

S. B. Falkovich, *Moscow, U.S.S.R.*

Th. von Kármán, *Columbia University, New York, New York*

G. Kuerti, *Harvard University, Cambridge, Massachusetts*

C. C. Lin, *Massachusetts Institute of Technology, Cambridge, Massachusetts*

P. F. Neményi, *Naval Research Laboratory, Washington, D. C.*

P. Ya. Polubarinova-Kochina, *Corresponding Member of the Academy of Sciences of the U.S.S.R., Moscow, U.S.S.R.*

E. Reissner, *Massachusetts Institute of Technology, Cambridge, Massachusetts*

PREFACE

This is the second volume of "Advances in Applied Mechanics" and follows the same principles as those indicated in the preface to Volume I. Due to various circumstances a longer interval than originally intended has elapsed between the publications of these first issues. It is hoped, however, that the third volume will appear within a shorter interval.

As has been stressed in the preface to the first volume, the principal aim of the "Advances" is to give surveys of the present state of research work in various fields of applied mechanics. The emphasis is not on giving abstracts of papers previously published nor is it intended to supply in each case a complete set of references to papers elsewhere accessible. Each article is written by an author who is actively engaged in the particular field of research and who has full liberty to develop his personal views and to present his own solutions. We think that in this way the reader is better served than by the routine type of summary or review.

Contributions to the *Advances* are, in general, by invitation of the Editors. Suggestions on topics to be dealt with and offers of special contributions will be appreciated and will receive careful consideration.

TH. v. KÁRMÁN
R. v. MISES

CONTENTS

Contributors to Volume II v
Preface vii

On the Statistical Theory of Isotropic Turbulence

By Th. von Kármán and C. C. Lin, *Columbia University, New York, New York, and the Massachusetts Institute of Technology, Cambridge, Massachusetts*

I. Relation between Correlation and Spectral Theories 2
II. Consideration of Similarity 6
III. Proposed Theory 11
IV. Detailed Study of the Early Period 16
V. Conclusion 18
References 19

The Laminar Boundary Layer in Compressible Flow

By G. Kuerti, *Harvard University, Cambridge, Massachusetts*

I. Introduction 23
II. The Basic Equations 23
III. The Simple Problem 29
IV. The Mathematics of Boundary Layer Theory 37
V. Results 69
References 91

Bending of Curved Tubes

By R. A. Clark and E. Reissner, *Massachusetts Institute of Technology, Cambridge, Massachusetts*

I. Introduction 93
II. Formulation of the Problem 97
III. The Equations for Bending of a Tube with Uniform Circular Cross Section 99
IV. Trigonometric Series Solution for the Tube with Uniform Circular Cross Section 102
V. Asymptotic Solution for the Tube with Uniform Circular Cross Section . . 109
VI. The Equations for Bending of a Tube with Uniform Elliptical Cross Section 117
VII. Asymptotic Solution for the Tube with Uniform Elliptical Cross Section . 118
References 122

Recent Developments in Inverse and Semi-Inverse Methods in the Mechanics of Continua

By P. F. Neményi, *Naval Research Laboratory, Washington, D. C.*

I. Introduction 123
II. Motion of Incompressible Inviscid Fluids 124
III. Flow of a Viscous Incompressible Fluid 126
IV. Gas Flows 130
V. Elastostatics 136
VI. Plasticity 141
VII. Significance and Heuristic Value of the Inverse and Semi-Inverse Methods of Mechanics 145
References 148

Theory of Filtration of Liquids in Porous Media

By P. Ya. Polubarinova-Kochina, *Corresponding Member of the Academy of Sciences of the U.S.S.R., Moscow, U.S.S.R.*, and S. B. Falkovich, *Moscow, U.S.S.R.*

I. Fundamental Equations of Steady Motion 154
II. Motion of Soil Waters without a Free Surface 157
III. Motions with a Free Surface—Zhukovsky's Function 163
IV. Motion with a Free Surface. Velocity Hodograph 165
V. The Seepage Line 171
VI. General Case of the Velocity Hodograph 175
VII. Application of the Analytic Theory of Differential Equations 176
VIII. Inverse Methods 182
IX. Other Methods 187
X. Filtration in Multilayer Soils 193
XI. Filtration of Oil 201
XII. Three-Dimensional Problems 206
XIII. Unsteady Flows 210
References 217

Author Index 227

Subject Index 231

On the Statistical Theory of Isotropic Turbulence[1]

By Th. von Kármán[2] and C. C. Lin[3]

Columbia University, New York, New York, and the Massachusetts Institute of Technology, Cambridge, Massachusetts

CONTENTS

I. Relation between Correlation and Spectral Theories 2
II. Consideration of Similarity 6
III. Proposed Theory 11
  1. The Early Stage 13
  2. The Intermediate Stage 15
IV. Detailed Study of the Early Period 16
V. Conclusion 18
  References 19

Much recent work has been done in the study of isotropic turbulence, particularly from the point of view of its spectrum. But the underlying concept is still the assumption of the similarity of the spectrum during the process of decay, which is equivalent to the idea of self-preservation of the correlation functions introduced by the senior author. It is, however, generally recognized that the correlation function does change its shape during the process of decay, and hence the concept of self-preservation or similarity must be interpreted with suitable restrictions. Under the limitation to low Reynolds numbers of turbulence, the original idea of Kármán-Howarth has been confirmed. Then the decay consists essentially in viscous dissipation of energy separately in each individual frequency interval. However, when turbulent diffusion of energy, i.e., transfer of energy between frequency intervals, occurs at a significant rate, the interpretation of the decay process and the spectral distribution is quite varied. This can be seen by a comparison of the recent publications of Heisenberg (3), Batchelor (4), Frenkiel (5), and the present authors (6,7). The purpose of the present paper is an attempt to clarify this situation.

Since some of these discussions are presented in terms of the correlation functions and others in terms of the spectrum, we shall begin by

[1] Invited paper at joint meeting of Fluid Mechanics Division, American Physical Society and Institute of Aeronautical Sciences, January 26, 1949. An abbreviated version of this paper appeared in the *Revs. Mod. Phys.*, 21, 516–519 (1949).

[2] Honorary professor, Columbia University.

[3] The work of Lin was partly sponsored by the Naval Ordnance Laboratory.

giving a simple systematic demonstration of the relation between the correlation theory and the spectral theory (Section I). This will serve as a background for the later developments. In Section II, we discuss the various considerations of similarity, their mutual relations, and a general comparison of their consequences with experiments. In Section III we propose a simple analysis of the spectrum, which gives a unified point of view regarding the types of similarity that should prevail at the various stages of the decay process. A detailed study of the early period of decay is given in Section IV.

It is concluded that the balance of energy among the fluctuations of various frequencies leads to the following conditions. At an early stage, all but the largest eddies participate in an equilibrium. There is a similarity of the spectrum if the lowest frequencies are excluded. In terms of correlation functions, this is the assumption proposed by the junior author (7). At an intermediate stage (to which the work of the senior author applies), the low-frequency components share the equilibrium of the largest eddies, and the high-frequency components form a separate regime of equilibrium. This separate high-frequency range is in general clearly defined for sufficiently high Reynolds number of turbulence, according to the concept of Kolmogoroff. Finally, when the Reynolds number becomes very low, the high-frequency components also share the equilibrium of the lowest frequencies and complete similarity is realized. There is then practically a separate viscous dissipation in each individual frequency range. When the initial Reynolds number is very high, the intermediate stage is very long. When the initial Reynolds number is very low, the intermediate stage practically disappears. Thus, to check the theory, experiments should be carried out at very high initial Reynolds number of turbulence, and the flow should be allowed to decay until the Reynolds number becomes very low. So far, no such experimental data seem to be available. Some suggestions for experimental measurements are made in Section V.

I. Relation between Correlation and Spectral Theories

It has been shown by von Kármán and Howarth that the change of the double correlation tensor R_{ik} is governed by the relation

$$\frac{\partial}{\partial t}(u'^2 R_{ik}) - u'^3 \frac{\partial}{\partial \xi_j}(T_{ijk} + T_{kji}) = 2\nu u'^2 \nabla^2 R_{ik} \tag{1.1}$$

where u'^2 is the mean square of the turbulent velocity, t is the time, ν is the kinematic viscosity coefficient, and $R_{ik}(\xi_l,t)$ and $T_{ijk}(\xi_l,t)$ are the double and triple correlation tensors for two points P and P' separated by a spatial vector ξ_l:

$$\begin{aligned} \overline{u_i(P)u_k(P')} &= u'^2 R_{ik} \\ \overline{u_i(P)u_j(P)u_k(P')} &= u'^3 T_{ijk} \end{aligned} \tag{1.2}$$

The above equation has been derived for homogeneous and isotropic turbulence, but it is not difficult to extend it to homogeneous anisotropic turbulence.

Another approach to the theory of turbulence is to make a Fourier analysis of the turbulent fluctuations. One may introduce this through the correlation theory by the spectral tensor $\mathfrak{F}_{ik}(\kappa_l)$ defined by the pair of Fourier transform relations:

$$\begin{aligned} \mathfrak{F}_{ik}(\kappa_l) &= \frac{1}{(2\pi)^3} \int\!\!\int\!\!\int u'^2 R_{ik}(\xi_m) e^{i(\kappa_l \xi_l)} d\tau(\xi_m) \\ u'^2 R_{ik}(\xi_l) &= \int\!\!\int\!\!\int \mathfrak{F}_{ik}(\kappa_m) e^{-i(\kappa_l \xi_l)} d\tau(\kappa_m) \end{aligned} \tag{1.3}$$

It is clear from the second equation that $\frac{1}{2}\mathfrak{F}_{nn}$ represents the density of energy u'^2 in the κ-space at frequency κ_i. One may now obtain an equation for the change of the spectrum $\mathfrak{F}_{nn}$ by a Fourier transform of (1.1). To complete the calculation one has to introduce the Fourier transform $\mathfrak{W}_{ijk}$ of the triple correlation tensor T_{ijk}:

$$\begin{aligned} \mathfrak{W}_{ijk}(\kappa_l) &= \frac{1}{(2\pi)^3} \int\!\!\int\!\!\int u'^3 T_{ijk}(\xi_m) e^{i(\kappa_l \xi_l)} d\tau(\xi_m) \\ u'^3 T_{ijk}(\xi_l) &= \int\!\!\int\!\!\int \mathfrak{W}_{ijk}(\kappa_m) e^{-i(\kappa_l \xi_l)} d\tau(\kappa_m) \end{aligned} \tag{1.4}$$

A simple calculation then shows that (1.1) becomes

$$\frac{\partial \mathfrak{F}_{ik}}{\partial t} + \mathfrak{W}_{ik} = -2\nu\kappa^2 \mathfrak{F}_{ik} \tag{1.5}$$

where

$$\mathfrak{W}_{ik} = i\kappa_j(\mathfrak{W}_{ijk} + \mathfrak{W}_{kji}) \tag{1.6}$$

Each of the equations (1.1) and (1.5) consists of six equations. These are in general dependent on the coordinate system chosen. Invariant relations are obtained by a contraction, and equation (1.5) then becomes an equation for the change of energy spectrum.

Further results on the spectral tensor have been obtained by J. Kampé de Fériet (8) in the case of homogeneous anisotropic turbulence; we shall however limit ourselves to the case of homogeneous isotropic turbulence. Here von Kármán and Howarth have shown that, by virtue of the incompressibility of the fluid, there is only one scalar correlation function $f(r,t)$ needed for characterizing R_{ik} and another correlation

function $h(r,t)$ for T_{ijk}. Hence, equation (1.1) is essentially given by a single equation, which has been shown to be

$$\frac{\partial}{\partial t}(u'^2 f) + 2u'^3\left(\frac{\partial h}{\partial r} + \frac{4h}{r}\right) = 2\nu u'^2\left(\frac{\partial^2 f}{\partial r^2} + \frac{4}{r}\frac{\partial f}{\partial r}\right) \tag{1.7}$$

It is obvious that this equation can be obtained by a contraction of (1.1). Similarly, in the spectral representation, one scalar function, such as $\mathfrak{F}_{kk}$, is able to characterize the whole spectral tensor $\mathfrak{F}_{ik}$; and the energy equation, obtained from a contraction, gives all the information contained in (1.5). This equation is more conveniently expressed in terms of the spectral function

$$\mathfrak{F} = \frac{4\pi\kappa^2}{3}\mathfrak{F}_{nn}, \tag{1.8}$$

since the second equation of (1.3) shows that

$$u'^2 = \int_0^\infty \mathfrak{F}(\kappa,t)d\kappa \tag{1.9}$$

If one further introduces the notation

$$\mathfrak{W} = \frac{4\pi\kappa^2}{3}\mathfrak{W}_{nn} \tag{1.10}$$

by analogy with (1.8), the final equation for energy transfer is then

$$\frac{\partial \mathfrak{F}}{\partial t} + \mathfrak{W} = -2\nu\kappa^2\mathfrak{F} \tag{1.11}$$

The relation between this equation and the Kármán-Howarth equation is not hard to find. In fact, if $\mathfrak{F}$ and $\mathfrak{W}$ are calculated with the help of the first formulas of (1.3) and (1.4), straightforward evaluation of the integrals by the use of spherical coordinates in the ξ-space gives

$$\begin{gathered}
\mathfrak{F} = \frac{1}{3}\{\kappa^2\mathfrak{F}_1''(\kappa) - \kappa\mathfrak{F}_1'(\kappa)\}, \\
\mathfrak{F}_1(\kappa) = \frac{2u'^2}{\pi}\int_0^\infty f(r)\cos\kappa r dr, \quad f(r) = \int_0^\infty \mathfrak{F}_1(\kappa)\cos\kappa r d\kappa \\
\mathfrak{W} = \frac{2\kappa^2}{3}\{\kappa^2 H_1''(\kappa) - \kappa H_1'(\kappa)\}, \\
\kappa H_1(\kappa) = \frac{2u'^3}{\pi}\int_0^\infty h(r)\sin\kappa r dr, \; h(r) = \int_0^\infty \kappa H_1(\kappa)\sin\kappa r d\kappa
\end{gathered} \tag{1.12}$$

These formulas for $\mathfrak{F}$ and $\mathfrak{W}$ have been obtained by Lin (9) by a somewhat different approach. The above discussion is essentially the one made

by Batchelor (10). It is clear from this kind of discussion that the transfer of energy among the various frequencies is independent of the pressure fluctuations.

The spectral equation (1.11) merely gives a different aspect of the theory without removing the fundamental difficulty inherent in this kind of approach to the theory of turbulence. There are still two functions occurring in (1.11): $\mathfrak{F}(\kappa,t)$ and $\mathfrak{W}(\kappa,t)$. To resolve this difficulty, one must try to establish some relation between the transfer function $\mathfrak{W}(\kappa,t)$, the spectrum $\mathfrak{F}(\kappa,t)$, and the frequency κ. The senior author proposed the relation

$$\int_0^\kappa \mathfrak{W}(\kappa,t)d\kappa = C\left\{\int_\kappa^\infty \{\mathfrak{F}(\kappa')\}^\alpha \kappa'^\beta d\kappa'\right\}\left\{\int_0^\kappa \{\mathfrak{F}(\kappa')\}^{3/2-\alpha}\kappa'^{1/2-\beta}d\kappa'\right\} \quad C > 0 \tag{1.13}$$

where α and β are constants. This includes Heisenberg's form

$$\int_0^\kappa \mathfrak{W}(\kappa,t)d\kappa = C\left\{\int_\kappa^\infty \{\mathfrak{F}(\kappa')\}^{1/2}\kappa'^{-3/2}d\kappa'\right\} \cdot \left\{\int_0^\kappa \mathfrak{F}(\kappa')\kappa'^2 d\kappa'\right\} \tag{1.14}$$

when $\alpha = \frac{1}{2}$, $\beta = -\frac{3}{2}$, and a modified Obukhoff's form

$$\int_0^\kappa \mathfrak{W}(\kappa,t)d\kappa = C\left\{\int_\kappa^\infty \mathfrak{F}(\kappa')d\kappa'\right\}\left\{\int_0^\kappa \{\kappa'\mathfrak{F}(\kappa')\}^{1/2}d\kappa'\right\} \tag{1.15}$$

when $\alpha = 1$, $\beta = 0$. It is essential from dimensional considerations that $\mathfrak{W}(\kappa,t)$ must be of degree three-halves in both $\mathfrak{F}$ and κ.

With these assumptions, the future behavior of the spectrum is completely known, once it is given at a certain instant. In the present paper, we shall however not pursue this line of attack any farther. We shall turn back to more general considerations, such as the similarity of turbulent fields.

A very important step in the study of the spectrum has been taken by Kolmogoroff. Whereas the previous similarity considerations postulated that the correlation functions and therefore also the spectral functions be expressed by u' as characteristic velocity and a characteristic length for the scale, Kolmogoroff postulates that at least at sufficiently high Reynolds numbers of turbulence, the local turbulence shall depend only upon the kinematic viscosity ν of the fluid and the total rate of energy dissipation

$$\epsilon = -\frac{du'^2}{dt} = 2\nu\int_0^\infty \kappa^2\mathfrak{F}(\kappa,t)d\kappa \tag{1.16}$$

In other words, the characteristic velocity and length of the turbulent state at any instant shall be fully determined by these two parameters. It then appears from purely dimensional arguments that in such a range,

the characteristic velocity and scale are

$$v = (\nu\epsilon)^{1/4}, \qquad \eta = (\nu^3/\epsilon)^{1/4} \tag{1.17}$$

If one further assumes that when $\kappa\eta \ll 1$, the spectrum is independent of ν, it then follows that

$$\mathfrak{F} = \text{constant } \epsilon^{2/3}\kappa^{-5/3} \tag{1.18}$$

which may be obtained from purely dimensional arguments and as a matter of fact was found by Obukhoff and developed independently by Onsager (11) and Weizsäcker (12). Kolmogoroff applied his postulate to the correlation function and concluded that

$$f(r,t) = 1 - \text{constant } u'^{-2}(\epsilon r)^{2/3} \tag{1.19}$$

in the limit of infinite Reynolds number. Equations (1.18) and (1.19) are essentially equivalent.

Another general result is due to Loitsiansky (13). He noted that the Kármán-Howarth equation (1.7) can be integrated to give the invariant relation

$$u'^2 \int_0^\infty f(r)r^4dr = \text{constant} \tag{1.20}$$

provided the integral is convergent and the triple correlation function vanishes so rapidly at infinity that $\lim_{r\to\infty} r^4h = 0$. It has been first shown by the junior author (9) with the help of equation (1.12) that this is equivalent to the conclusion that

$$\lim_{\kappa\to 0} \frac{\mathfrak{F}}{\kappa^4} = J_0, \quad \text{a constant} \tag{1.21}$$

This relation has also been found independently by Batchelor (10).

The results of Kolmogoroff and Loitsiansky will play an important role in the developments of the present paper; but we shall chiefly be guided by the idea of self-preserving correlation functions, or what is equivalent, the similarity of the spectrum.

II. Consideration of Similarity

The energy distribution among the various frequencies is changing through the transfer mechanism and through the effect of viscous dissipation [equation (1.11)]. It is reasonable to expect that the exchange of energy among the various frequencies will cause the spectrum of turbulence to tend to be similar in the course of time. This is the equivalent to the idea of self-preserving correlation functions introduced by von Kármán.

Let us consider the equation for the change of spectrum

$$\frac{\partial \mathfrak{F}}{\partial t} + \mathfrak{W} = -2\nu\kappa^2\mathfrak{F} \tag{2.1}$$

and try to find a similarity solution. If V is a characteristic velocity, and l is a characteristic length, then, from dimensional arguments.[4]

$$\mathfrak{F} = V^2 l \psi(\xi), \qquad \mathfrak{W} = V^3 w(\xi), \quad \xi = \kappa l \tag{2.2}$$

Thus, (2.1) becomes

$$\frac{1}{V}\frac{dl}{dt}\{\xi\psi'(\xi) + \psi(\xi)\} + \frac{2l}{V^2}\frac{dV}{dt}\psi(\xi) + w(\xi) = -\frac{2\nu}{Vl}\xi^2\psi(\xi) \tag{2.3}$$

If the similarity solution is to be valid, one must have

$$\frac{1}{V}\frac{dl}{dt} = a_1, \quad \text{a constant} \tag{2.4}$$

$$\frac{2l}{V^2}\frac{dV}{dt} = a_2, \tag{2.5}$$

$$\frac{\nu}{Vl} = a_3 \tag{2.6}$$

and the equation (2.3) becomes

$$a_1\xi\psi'(\xi) + (a_1 + a_2)\psi(\xi) + 2a_3\xi^2\psi(\xi) + w(\xi) = 0 \tag{2.7}$$

Besides (2.4)–(2.6) it is evident that the quantities u'^2 and ϵ, according to the definitions (1.9) and (1.16) have to satisfy the relations

$$u'^2 = V^2 \int_0^\infty \psi(\xi)d\xi \tag{2.8}$$

$$-\frac{du'^2}{dt} = \epsilon = 2\nu\frac{V^2}{l^2}\int_0^\infty \xi^2\psi(\xi)d\xi \tag{2.9}$$

Finally if the convergence criteria for Loitsiansky's relation (1.21) are assumed to be valid, we have

$$V^2 l^5 \lim_{\xi\to 0}\left\{\frac{\psi(\xi)}{\xi^4}\right\} = J_0 \tag{2.10}$$

This system of equations presumes that the transfer term in (2.3) is considered generally of equal importance with the term expressing the viscous dissipation. It has been shown by Dryden (14) in the equivalent

[4] This type of discussion is used in a memorandum by Lin to the Mechanics Division of the Naval Ordnance Laboratory during the spring of 1948. Abstract of paper appeared in the *Bull. Am. Math. Soc.*, 55, 49 (1949). (Submitted June 28, 1948.)

problem of self-preserving correlation functions that such a solution is connected with the statement that the square of the characteristic length is proportional to the time t and the law of decay is expressed by $u'^2 \sim t^{-1}$. Heisenberg (3) indicated in a recent paper an equivalent solution for the spectral problem. It is easily seen that Dryden's and Heisenberg's solutions are at variance with equation (2.10). In other words, *full similarity is only possible when we reject Loitsiansky's theorem.* This fact has been noted by Heisenberg (3). In addition, experimental evidence clearly indicates that the law of decay and the behavior of the characteristic length during decay exclude the possibility to adopt full similarity as a *general valid assumption.*

Let us consider now two opposite specific cases: (*a*) that the transfer term is considered negligible for all frequencies; (*b*) that the influence of viscous dissipation is restricted to high frequencies, whereas for low frequencies the transfer term is the prevailing factor.

Case (*a*), $w(\xi) = 0$, leads to a solution of equation (2.1) which furnishes full similarity for all frequencies and also satisfies Loitsiansky's relation. One obtains with $\xi = \kappa l$ and $l = \sqrt{\nu t}$

$$\mathfrak{F} = \text{constant } V^2 l \xi^4 e^{-2\xi^2} \tag{2.11}$$

or

$$\mathfrak{F} = \text{constant } V^2 l^5 \kappa^4 e^{-2\kappa^2 \nu t}. \tag{2.12}$$

By the definition of J_0 in (1.21) we write

$$\mathfrak{F} = J_0 \kappa^4 e^{-2\nu\kappa^2 t} \tag{2.13}$$

The corresponding correlation function can be easily shown to be

$$f(r,t) = e^{-r^2/8\nu t} \tag{2.14}$$

by using the first set of formulas in (1.12). This correlation function was noted by von Kármán and Howarth, and discussed by Millionshchikov (15), Loitsiansky (13), and Batchelor and Townsend (16). The law of decay in this case is the five-fourths power law:

$$u'^2 \sim (t - t_0)^{-5/4}, \quad \lambda^2 = 4\nu(t - t_0) \tag{2.15}$$

This law of decay and the corresponding correlation function has been verified experimentally by Batchelor and Townsend (16) for the final stage of decay.

It is generally believed that this solution furnishes a fair approximation for the final stage of the decay in any case and it is believed that it is correct for almost the whole process of decay if the initial Reynolds number of the turbulent motion is very small.

Case (*b*) has also been treated in the theory of self-preserving correlations by von Kármán and Howarth and later by Kolmogoroff (17). The former authors came to the conclusion that any power law for the decay-time relation may prevail in the decay process. Kolmogoroff pointed out that if one assumes the validity of Loitsiansky's theorem, the relations

$$u'^2 = \text{constant}\, t^{-10/7} \quad \text{and} \quad \lambda^2 = 7\nu t \tag{2.16}$$

must apply.

The senior author dealt with the corresponding spectral problem in two communications (6) assuming the specific decay law (2.16). He obtained results for the spectral distribution which are in good accordance with measurements. Especially it appears that the spectral distribution has the form $\mathfrak{F} \sim \kappa^4$ for small values of κ and $\mathfrak{F} \sim \text{constant}\, \kappa^{-5/3}$ for large values of κ provided in the whole κ-range considered the viscous dissipation remains small. We shall therefore refer to this case as the case with nonviscous similarity..

The junior author (7) was led by his consideration of general similarity to a law of decay of the form

$$u'^2 = a(t - t_0)^{-1} + b \tag{2.17}$$

where a is a positive and b a negative constant.

Subsequent discussion between the authors resulted in an analysis which is believed to clarify the situation essentially and constitute the main substance of this paper. It requires the introduction of two elements characterizing the process of decay: the time history and the initial Reynolds number.

We consider now as *case* (*c*) the assumption that similarity extends over the whole frequency range, with the exception of the lowest. In order to satisfy Loitsiansky's relation we assume that $\mathfrak{F}(\kappa)$ behaves for small κ as proportional to κ^4.

The assumption is essentially the following. The deviation from similarity shall occur only for such small values of κ that whereas the contribution of the deviation is negligible for computation of ϵ [cf. equation (2.9)], it enters in the calculation of the energy [cf. (2.8)].

It appears that this assumption is equivalent to Kolmogoroff's similarity hypothesis in Lin's extended formulation. By using the first set of formulas in (1.12) one can easily verify that

$$u'^2(1 - f) = C_2 \frac{r^2}{2!} - C_4 \frac{r^4}{4!} + \cdots \tag{2.18}$$

where

$$C_{2n} = \frac{1}{(2n + 1)(2n + 3)} \int_0^\infty \mathfrak{F}(\kappa,t)\kappa^{2n} d\kappa \tag{2.19}$$

Thus, if the deviation from similarity has a negligible effect in ϵ, i.e., C_2, then C_{2n}'s are all proportional to $V^2 l^{-2n}$. This is the assumption made in Lin's paper. Conversely, if the similarity of $u'^2(1 - f)$ is assumed, the spectrum is similar to the extent that the moments C_{2n} are all proportional to $V^2 l^{-2n}$.

The law of decay can be obtained from the general relations (2.4), (2.5), (2.6), and (2.9), which are valid for any similarity hypothesis. One obtains the positive and negative half-power laws for the change of the characteristic length l and the characteristic velocity V, and the inverse square law for the rate of dissipation ϵ. To be more specific, one may identify l and V with Kolmogoroff's characteristic quantities [cf. (2.6) and (2.9)]

$$\eta = (\nu^3/\epsilon)^{1/4} \quad \text{and} \quad v = (\nu\epsilon)^{1/4} \tag{2.20}$$

It can be easily seen by introducing these relations into (2.4)–(2.6), that the law of decay is of Lin's form (2.17).

For convenience of reference, the results are listed below with definite physical interpretation attached to the constants. The law of decay is given by

$$\epsilon = (D_0/10)t^{-2}, \qquad u'^2 = (D_0/10)t^{-1} - u_D^2, \qquad \lambda^2 = 10\nu t\left(1 - \frac{10u_D^2}{D_0}t\right) \tag{2.21}$$

where $u_D{}^2$ is the additive constant giving the departure from similarity, and D_0 is the initial diffusion coefficient

$$D_0 = \lim_{t\to 0} \frac{u'^2\lambda^2}{\nu} \tag{2.22}$$

defined according to a formula of the kind suggested by von Kármán (1) in 1937. The change of the characteristic velocity and scale, and of the Reynolds number of turbulence are given by

$$v^2 = (10)^{-1/2}R_{\lambda 0}\nu t^{-1}, \qquad \eta^2 = (10)^{1/2}R_{\lambda 0}^{-1}\nu t, \qquad R_\lambda = R_{\lambda 0}\left(1 - \frac{10u_D^2}{D_0}t\right) \tag{2.23}$$

where $R_{\lambda 0}$ is the initial Reynolds number of turbulence,

$$R_{\lambda 0} = \lim_{t\to 0} \frac{u'\lambda}{\nu} \tag{2.24}$$

It is evident from the equations (2.21) and (2.23) that the solutions obtained can only be applied to an early stage of the decay process, in which $10u_D^2 t/D_0$ remains small.

Most experiments on decay have been carried out for relatively small

values of $R_{\lambda 0}$. They confirm the conclusions of Lin's theory at least for the early stage of decay. In particular, the linear decrease of R_λ is indicated. Also the few experiments where $R_{\lambda 0}$ is not small, show that in the initial stage the relation $\lambda^2 = 10\nu t$ approximately holds. Therefore although no a priori reason can be found why the decay should not start according to the law specified by the nonviscous similarity theory, we propose that for the early stage the hypothesis of case (*c*) holds, whatever the initial Reynolds number may be.

In the next section a general discussion is given covering the whole process of decay.

III. Proposed Theory

The linear decrease of R_λ as given by Lin's theory cannot last very long, because the slope of the $(\lambda^2, \nu t)$ curve is given by $10[2R_\lambda/R_{\lambda 0} - 1]$ and therefore when R_λ becomes equal to $17R_{\lambda 0}/20$, the law of decay is already similar to that governed by the nonviscous kind of similarity. [Compare (2.16) and (2.23).] One may reasonably suppose that this kind of similarity then actually prevails. The physical basis for this change of nature of similarity will become clear after an analysis of the change of spectrum—as we shall do later.

During this period of nonviscous similarity, the Reynolds number R_λ keeps on decreasing and eventually becomes so small that the final period of decay is reached. If the initial Reynolds number $R_{\lambda 0}$ is very large, the time required for this change from $17R_{\lambda 0}/20$ to small Reynolds numbers is very long. One would estimate a very long nonviscous period in which $\lambda^2 = 7\nu t$. This has never been thoroughly investigated experimentally: most of the measurements with high Reynolds numbers of turbulence scarcely extend beyond the early period. On the other hand, those experiments which do cover both the early and the final periods are performed at such small Reynolds numbers, that a complete omission of the range of nonviscous similarity gives a rather good description of the decay process.

We shall therefore devote our later discussions to cases where the initial Reynolds number is large. We shall consider the spectrum and the law of decay during the various stages in the idealized case of an infinite field of turbulence. In comparing the theory with experimental investigations, one must bear in mind that the actual size of the apparatus may be a limitation to the behavior of the largest eddies.

First we note that Kolmogoroff's theory as discussed above, gives a definite meaning to the high-frequency components, at least for large Reynolds numbers. It is the range in which viscous effects are dominating and the major part of dissipation is occurring. On the other hand,

it contains a negligible fraction of the energy. The spectrum in this range depends on ϵ and ν.

The part of the spectrum below this range is related to the large eddies which are the carriers of turbulent diffusion. It should be possible to characterize the spectrum in this range by a turbulent diffusion coefficient D. It includes the spectrum for very small values of κ which, as shown above, has the form $J_0\kappa^4$ and therefore is practically invariant in time. Then it includes those somewhat smaller eddies which significantly decrease in intensity, since they supply energy to the high-frequency range to be dissipated at the rate ϵ. We shall therefore distinguish again two frequency ranges: (1) the largest eddies characterized by J_0 and D, (2) the medium frequency range characterized by D and ϵ. In other words, if V^* and L^* are the characteristic velocity and length of the largest eddies, and V and L are corresponding quantities for the medium eddies, we are assuming that

$$VL = V^*L^* \tag{3.1}$$

which quantity is then regarded as the turbulent diffusion coefficient D, characteristic for low-frequency phenomena.

We have thus divided the spectrum into three ranges: (1) the high-frequency range, depending on ϵ and ν; (2) the low-frequency range, depending on ϵ and D; (3) the range of largest eddies, depending on D and J_0. By dimensional arguments, one can obtain the characteristic quantities and other general properties of the spectrum in the various ranges. These are tabulated in Table I. The formulas for the spectrum

TABLE I

Frequency range	(1) High	(2) Medium	(3) Low
Basic characteristic quantities	ν, ϵ	ϵ, D	D, J_0
Length	$\eta = (\nu^3/\epsilon)^{1/4}$	$L = (D^3/\epsilon)^{1/4}$	$L^* = (J_0/D^2)^{1/3}$
Velocity	$v = (\nu\epsilon)^{1/4}$	$V = (D/\epsilon)^{1/4}$	$V^* = (D^5/J_0)^{1/3}$
Time	$\tau = (\nu/\epsilon)^{1/2}$	$T = (D/\epsilon)^{1/2}$	$T^* = (J_0^2/D^7)^{1/3}$
Form of spectrum	$v^2\eta\phi(\kappa\eta)$	$V^2LF(\kappa L)$	Fixed $V^{*2}L^*F_1(\kappa L^*)$
lower end	$\mathfrak{F} =$ const. $\epsilon^{2/3}\kappa^{-5/3}$	$\mathfrak{F} =$ const. $D^2\kappa$	$\mathfrak{F} = J_0\kappa^4$

in the transition ranges can be obtained from purely dimensional arguments. They can exist only when the scales η, L, and L^* are very much different from each other. To examine the conditions required, let us note that

$$\frac{\eta}{L} = \left(\frac{D}{\nu}\right)^{-3/4}$$
$$\frac{L}{L^*} = \left(\frac{T}{T^*}\right)^{1/2} \tag{3.2}$$

We shall see that these conditions show that the $\kappa^{-5/3}$-range would exist if the Reynolds number R_λ is large, and that the linear range will exist during the early part of the history of decay.

The above analysis of the spectrum enables us to give a very simple *physical description of the process of decay*. We divide the process into three stages: (1) the early stage, (2) the intermediate stage, and (3) the final stage, in which the Reynolds number is small. This last stage has been fully clarified before and will not be discussed here.

1. The Early Stage

At the beginning of the process of decay, the small eddies, range (1), first reached a state of equilibrium. To this the low-frequency components, range (2), may be expected to participate through the mechanism of energy transfer. The ratios η/L, v/V, τ/T must therefore be constants. This is the kind of similarity discussed as case (*c*) of Section II. The law of decay is characterized by (2.21) and (2.23). One can easily verify from (2.23) that the diffusion coefficient

$$D = VL \sim v\eta \tag{3.3}$$

is proportional to D_0. We shall therefore identify D and D_0 since both are a measure of diffusion. This means that V^* and L^* are fixed, since on the one hand we have $V^{*2}L^{*5} = J_0$, and on the other hand $V^*L^* = D_0$. The large eddies are invariant up to the linear part of spectrum.

The physical interpretation of the additive constant u_D^2 in (2.23) is now clear. From the above discussions, the spectrum must be of the type shown in Fig. 1. The part of the spectrum above the linear range shrinks radially inversely proportional to the square root of time [cf. (2.23) and the formula for the spectrum in Table I]. The linear part and the lower frequencies are fixed. The additive constant is simply the shaded area, representing the deviation of the lowest frequencies from complete similarity. This can be seen by calculating u'^2 by integration over the spectrum:

$$u'^2 = \int^{\infty} \mathfrak{F}(\kappa)d\kappa = V^2 \int_0^{\infty} \phi(\xi)d\xi - u_D^2 \tag{3.4}$$

which is the law of decay (2.17) first given by the junior author.

The role of these relatively invariant large eddies has been independently discussed by Heisenberg (3), Batchelor (4), and the junior author (7). In particular, Fig. 1 is implied in Heisenberg's discussion and has been shown explicitly by Batchelor.

One can easily verify, with $D = D_0$, that

$$\frac{\eta}{L} = \frac{v}{V} = \left(\frac{D}{\nu}\right)^{-3/4} = R_{\lambda 0}^{-3/2} \tag{3.5}$$

Comparing (3.2) with (3.5) this means that the $\kappa^{-5/3}$ range would exist when $R_{\lambda 0}$ is large, in agreement with the idea of Kolmogoroff. The second relation in (3.2), which is the condition for the existence of the

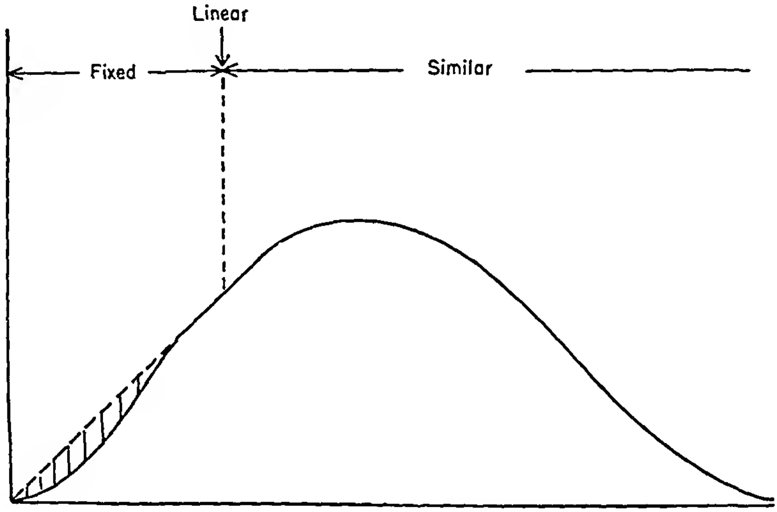


FIG. 1. Spectrum during early stage of decay process.

linear part of the spectrum, can also be easily worked out explicitly. One obtains

$$\frac{L}{L^*} = \left(\frac{\lambda^2}{\sqrt{10}\,\nu T^*}\right)^{1/2} \tag{3.6}$$

Hence, by noting the fact that $d\lambda^2/d(\nu t) \leq 10$, one can see that $L/L^* \ll 1$ essentially for the early period when t/T^* is small. That is why the above kind of similarity, which depends on the existence of the linear range of spectrum, can exist only during the early period. This is equivalent to the condition reached in Section II that $10u_D^2 t/D_0$ should remain small, as will be demonstrated more fully with an example in Section IV.

It may be noted that, in the above discussions, the existence of the $\kappa^{-5/3}$ range is not essential, and hence the Reynolds number need not be large. In fact, for small initial Reynolds numbers, the Reynolds number at the end of the early period may be so small that the final period already sets in. This explains the agreement obtained by the junior author in

comparing his theory with the experiments made by Batchelor and Townsend over the whole process of decay (7).

2. *The Intermediate Stage*

For large Reynolds numbers, after the disappearance of the linear part of the spectrum, the scales L and L_0 become of the same order of magnitude, and it may be expected that the bulk of turbulent energy of scale L shares the behavior of the large eddies of scale L^*. The ratios L/L^*, V/V^*, and T/T^* are expected to be constants. These conditions lead at once to the law of decay discussed by the senior author. It is characterized by $d\lambda^2/d(\nu t) = 7$.

During this period, the spectrum at low and medium frequencies depends only on the parameters J_0 and ϵ, and must therefore be of the form

$$J_0\kappa^4\Phi(\kappa L)$$

where $\Phi(\kappa L)$ behaves as $(\kappa L)^{-17/3}$ for $\kappa L \gg 1$, and approaches unity when $\kappa L \rightarrow 0$. An interpolation formula for Φ has been suggested and checked by correlation measurements by the senior author (6).

The diffusion coefficient in this range is easily seen to be proportional to $u'^2\lambda^2/\nu$. Hence, the condition (3.2) is again that the Reynolds number should be large. When the Reynolds number becomes very small, the scales η and L are of the same order, so that there is only one scale for all frequencies. We then approach a complete similarity and are at the beginning of the final period. With reference to (3.2), we see that it should happen when R_λ is of the order of unity. According to the experiments of Batchelor and Townsend (10), the final period sets in at $R_\lambda \sim 5$.

The intermediate stage is very long, if the initial Reynolds number is very large. It begins with some value of R_λ close to $R_{\lambda 0}$. During this period, R_λ changes according to the power law $t^{-3/14}$. Although the supposed origin of time in this formula is unknown, it must be before the beginning of the early period, since the slope of the λ^2 versus νt curve decreases. Thus, R_λ becomes of the order of unity only when t is of the order of $T^*R_{\lambda 0}{}^{14/3}$. One expects therefore to find an intermediate stage many times the early period for very high initial Reynolds numbers. In such a case, the final period could probably never be observed in practice. Such predictions can only be checked by experiments at very high Reynolds numbers performed over an extended period of decay. For although the initial period is relatively short, it extends over a very long distance in the wind tunnel when the wind speed is high.

IV. Detailed Study of the Early Period

It is instructive to work out in some detail the above general discussions during the early period, in the case of high initial Reynolds numbers. In particular, we shall demonstrate explicitly that u_D^2 is proportional to V^{*2}. We shall also demonstrate the precise meaning of the statement that at high Reynolds numbers of turbulence, the major part of energy lies in the low-frequency range and the major part of dissipation lies in the high-frequency range.

The spectrum in the early period is given in Table I, with all the three ranges present. The intermediate range with $\mathfrak{F} \sim \kappa^{-5/3}$ obtains when $x = \kappa\eta \ll 1$ and $X = \kappa L \gg 1$. Let us choose a frequency κ_2 somewhere in this range. Then

$$X_2 = \kappa_2 L \gg 1, \qquad x_2 = \kappa_2 \eta \ll 1 \tag{4.1}$$

Since the ratio x_2/X_2 is of the order of $R_{\lambda 0}^{-3/2}$ [see (3.5)], one may so choose κ_2 that X_2 is of the order of $R_\lambda^{3/4}$, say, and x_2 of the order of $R_{\lambda 0}^{-3/4}$. The numbers x_2 and X_2 are parameters depending only on $R_{\lambda 0}$ but invariant with time. Similarly, one may choose a frequency κ_1 in the linear range of the spectrum satisfying the conditions

$$X_1 = \kappa_1 L \ll 1, \qquad X^* = \kappa_1 L^* \gg 1 \tag{4.2}$$

The magnitude of a quantity here is, however, determined by the time history rather than the Reynolds number.

Our present interest centers around the rélative contribution to energy and dissipation in the high-frequency range as compared with those in all the other parts of the spectrum. Thus, the exact nature of the spectrum around the linear range is not too important; and for convenience, we shall approximate the spectrum by

$$\begin{aligned} \mathfrak{F}(\kappa,t) &= J_0\kappa^4 && \text{for } \kappa < \kappa_1 \\ &= V^2 L F(\kappa L) && \text{for } \kappa_1 < \kappa < \kappa_2 \\ &= v^2 \eta \phi(\kappa\eta) && \text{for } \kappa_2 < \kappa \end{aligned} \tag{4.3}$$

where κ_1 may be taken to be the intersection of $\mathfrak{F} = J_0\kappa^4$ with the linear law constant $D_0^2\kappa$. The function F and ϕ outside the range specified above will be defined by their limiting behavior. Thus, $F(X)$ is proportional to X for small X, and proportional to $X^{-5/3}$ for large X; and $\phi(x)$ is proportional to $x^{-5/3}$ for small x. The energy and its rate of dissipation are respectively given by

$$u'^2 = V^2 \int_0^{X_1} F(X)dX + v^2 \int_{x_1}^{\infty} \phi(x)dx - \frac{3}{10} J_0 \kappa_1^5 \tag{4.4}$$

$$\frac{\epsilon}{2\nu} = \frac{V^2}{L^2} \int_0^{X_1} F(X)X^2 dX + \frac{v^2}{\eta^2} \int_{x_1}^{\infty} \phi(x)x^2 dx - \frac{3}{28} J_0 \kappa_1^7$$

for the spectrum (4.3).

We want to show that, as expected, in the expression for the energy u'^2, the part

$$u_l'^2 = V^2 \int_0^{X_1} F(X)dX \tag{4.5}$$

related to the low-frequency components, is much larger than the part

$$u_\lambda'^2 = v^2 \int_{x_1}^{\infty} \phi(x)dx \tag{4.6}$$

related to the high-frequency components. On the other hand, in the expression for the dissipation $\epsilon/2\nu$, the part

$$\frac{\epsilon_l}{2\nu} = \frac{V^2}{L^2} \int_0^{X_1} F(X)X^2 dX \tag{4.7}$$

related to the low-frequency components, is much smaller than the part

$$\frac{\epsilon_\lambda}{2\nu} = \frac{v^2}{\eta^2} \int_{x_1}^{\infty} \phi(x)x^2 dx \tag{4.8}$$

related to the high-frequency components. In fact, we shall see that

$$\begin{aligned} u_l'^2 &= u_\lambda'^2 O(X_2^{2/3}) \\ \epsilon_l &= \epsilon_\lambda O(x_2^{4/3}) \end{aligned} \tag{4.9}$$

It is easily seen from direct calculation that

$$u_l'^2 = V^2 \left[\int_0^{\infty} F(X)dX + O(X_2^{-2/3}) \right] \tag{4.10}$$

and

$$u_\lambda'^2 = v^2 \left\{ C x_2^{-2/3} + \int_{x_1}^{\infty} [\phi(x) - \tfrac{2}{3} C x^{-5/3}] dx \right\}$$

where we have stipulated that $\phi(x) \sim \frac{2}{3} C x^{-5/3}$ for small x. The second term may consequently be approximated by the constant value

$$\int_0^{\infty} [\phi(x) - \tfrac{2}{3} C x^{-5/3}] dx$$

for small values of x_2, and is therefore much smaller compared with $C x_2^{-2/3}$. Now from Table I, we see that

$$\frac{v}{V} = \left(\frac{\eta}{L}\right)^{1/3} = \left(\frac{x}{X}\right)^{1/3} \tag{4.11}$$

Hence, the above expression for $u_h'^2$ may be written

$$u_h'^2 = CV^2X_2^{-2/3}[1 + O(x_2^{2/3})] \tag{4.12}$$

Equations (4.10) and (4.12) give the first relation of (4.9) desired. Indeed, by making use of (4.10) and (4.12), the energy expression (4.4) may be written as

$$u'^2 = (1 - \delta_1)V^2 \int_0^\infty F(X)dX - \tfrac{3}{10}J_0\kappa_1^5, \quad \delta_1 = O(X_2^{-2/3}) \tag{4.13}$$

Similar arguments lead to the second relation in (4.9) and indeed give

$$\frac{\epsilon}{2\nu} = (1 - \delta_2)\frac{v^2}{\eta^2}\int_0^\infty \phi(x)x^2dx - \frac{3}{28}J_0x_1^7, \quad \delta_2 = O(x_2^{2/3}) \tag{4.14}$$

Since $V^2 \sim t^{-1}$ and $v^2/\eta^2 \sim t^{-2}$, the additive constants in (4.13) and (4.14) are not important. For small but finite values of t, one should retain the additive constant in u'^2 but reject it in $\epsilon/2\nu$, because it is only part of the low frequency contribution, which is wholly negligible in the case of dissipation. Thus (4.12) and (4.14) agree with (2.21) if

$$(1 - \delta_1)\cdot\int_0^\infty F(X)dX = 1, \qquad (1 - \delta_2)\cdot\int_0^\infty \phi(x)x^2dx = \tfrac{1}{2}$$

and

$$u_D^2 = \tfrac{3}{10}J_0\kappa_1^5$$

Since $J_0\kappa_1^4 =$ constant $D_0^2\kappa_1$, it can be easily seen that (cf. Table I)

$$u_D^2 = \text{constant } V^{*2} \tag{4.15}$$

From this, it follows that the estimation of the initial period made in Sections II and III are identical.

V. Conclusion

The proposed simple physical picture described in Section III explains all experimental data so far available. However, experimental data are yet lacking regarding a transition from the law $\lambda^2 = 10\nu t$ to the law $\lambda^2 = 7\nu t$ during the early period. The experiments must be performed at very high Reynolds numbers over an extended period of decay.

More careful experimental examination of the variation of R_λ is also desired to see whether the law asserting its linear decrease in the early period is confirmed in all cases. According to the discussion of the junior author (7), the quantity $G = f^{iv}(0)\lambda^4$ should also decrease linearly in time, but the ratio G/R_λ should be constant in any given experiment. It varies with the initial Reynolds number according to the relation

$$\frac{G}{R_\lambda} = \frac{S}{2} + \frac{30}{7R_{\lambda 0}}$$

where $S = k'''(0)\lambda^3$ should be an absolute constant. The above formula has been verified by Batchelor and Townsend over a very limited period during which the variation of R_λ cannot be definitely established because of the experimental scatter.

The best experimental check would be a direct examination of the similarity hypotheses by measurements of the spectrum and the correlation functions. In particular, it has been asserted (Heisenberg) that the appearance of a linear part of the spectrum is associated with the early period but not related to the Reynolds number of turbulence, while the appearance of the five-thirds power range depends essentially on the instantaneous value of Reynolds number.

It must be kept in mind that all the above discussions hold only for an infinite field of turbulence. If the scale becomes comparable with that of the experimental apparatus, e.g., the cross-sectional dimension of the wind tunnel, one would be determining the influence of the wind tunnel instead of the natural properties of the turbulent field.

References

1. von Kármán, Th., *J. Aeronaut. Sci.*, **4**, 131–138 (1937).
2. von Kármán, Th., and Howarth, L., *Proc. Roy. Soc.* (*London*), **A164**, 192–215 (1938).
3. Heisenberg, W., *Proc. Roy. Soc.* (*London*), **A195**, 402–406 (1948).
4. Batchelor, G. K., "Recent developments on turbulence research," General lecture at the Seventh International Congress for Applied Mechanics, 1948.
5. Frenkiel, F. N., "Comparison between some theoretical and experimental results of the decay of turbulence," presented at the Seventh International Congress for Applied Mechanics, 1948.
6. von Kármán, Th., *Compt. rend.*, **226**, 2108–2110 (1948); *Proc. Nat. Acad. Sci.*, **34**, 531–539 (1948).
7. Lin, C. C., *Proc. Nat. Acad. Sci.*, **34**, 540–543 (1948); also "On the law of decay and the spectrum of isotropic turbulence," presented at the Seventh International Congress for Applied Mechanics, 1948.
8. Kampé de Fériet, J., *Compt. rend.*, **227**, 760–761 (1948).
9. Lin, C. C., "Remarks on the spectrum of turbulence," presented at the First Symposium of Applied Mathematics, AMS, 1947.
10. Batchelor, G. K., *Proc. Roy. Soc.* (*London*), **A195**, 513–532 (1949).
11. Onsager, L., *Phys. Rev.*, **68**, 286 (1945) (abstract only).
12. von Weizsäcker, C. F., *Z. Physik*, **124**, 614–627 (1948).
13. Loitsiansky, L. G., C.A.H.I. (Moscow), *Rept.* **440**, 1939; also N.A.C.A. *Tech. Memo.* **1079**.
14. Dryden, H. L., *Quart. Applied Math.*, **1**, 7–42 (1943).
15. Millionshchikov, M. D., *Compt. rend. acad. sci. U.R.S.S.*, **22**, 231–237 (1939).
16. Batchelor, G. K., and Townsend, A. A., *Proc. Roy. Soc.* (*London*), **194**, 527–543 (1948).
17. Kolmogoroff, A. N., *Compt. rend. acad. sci. U.R.S.S.*, **31**, 538–540 (1941).

The Laminar Boundary Layer in Compressible Flow

By G. Kuerti

Harvard University, Cambridge, Massachusetts

CONTENTS

I. Introduction.. 23
II. The Basic Equations.. 23
1. The Equations of the Viscous Compressible Fluid.................. 23
2. The Equations of the Boundary Layer.......................... 26
III. The Simple Problem.. 29
1. History and General Character of the Solution.................. 29
2. Heat Transfer Characteristics.......................... 32
IV. The Mathematics of Boundary Layer Theory.......................... 37
1. Transformations That Work under Suitable Simplifying Assumptions about the Parameters μ, ρ, and σ.......................... 37
A. The Independent Variable Ψ (v. Kármán and Tsien, Chapman and Rubesin)—The Viscosity Law.......................... 38
B. The Independent Variable U (Crocco, Hantzsche and Wendt)... 42
C. The Stretched y-Coordinate (Howarth).......................... 46
2. Transformations Adapting Compressible Boundary Layer Equations to Application of Approximate Methods Developed for Incompressible Boundary Layer.......................... 48
A. The Stretched y-Coordinate (Kalikhman)—Generalization of Pohlhausen's Method by Kalikhman.......................... 48
B. Generalization of Pohlhausen's Method by Howarth........... 55
3. Transformations That Prepare the Equations for a Numerical Computer 57
A. Series Expansions (Howarth).......................... 57
B. The Dependent Variable $\frac{1}{\rho}$; the General Problem of Series Expansion (Cope and Hartree).......................... 60
C. Numerical Solution of the Restricted Problem (Emmons and Brainerd). Asymptotic Integration (Meksyn)................ 63
V. Results.. 69
1. The Simple Problem.. 70
2. The Restricted Problem for $\omega = 1$.......................... 72
3. The Restricted Problem for $\omega \neq 1$.......................... 75
4. Other Problems.. 86
References.. 91

NOTATION

(Bracketed symbols are dimensionless)

C constant of proportionality in the linearized Sutherland formula, equation (4.13)

b	compressibility factor $(\gamma - 1)M^2$ or $U_\infty^2/(c_pT_\infty)$
c_p	specific heat at constant pressure (ft-lb/sl deg)
c_v	specific heat at constant volume (ft-lb/sl deg)
F [C_{D_f}]	friction drag force (note $C_{D_f} = 2c_{\tau_0}\sqrt{x/L}$)
$g(u)$	dimensionless stress variable, Hantzsche and Wendt, equation (4.41)
I	enthalpy per unit of mass (ft-lb/sl)
$K(u)$	dimensionless stress variable, Crocco, equation (4.30)
k, [$\bar{k}$]	coefficient of thermal conductivity (ft-lb/ft sec deg)
M, M_∞	Mach number along edge of boundary layer
Nu	Nusselt number $Q_w/(k_\infty \Delta T_{\text{representative}})$
p, [$\bar{p}$]	pressure
$\mathbf{q}$	velocity vector
q_w	heat transmitted to fluid per unit of time and area (ft-lb/ft^2 sec)
Q_w	$\int_0^L q_w dx$ (ft-lb/ft sec)
R	gas constant per unit of mass (ft-lb/sl deg)
Re	Reynolds number, mostly $U_\infty L/\nu_\infty$
T, [t]	absolute temperature
T^*, [t^*]	stagnation temperature, equation (3.3)
T_e, [t_e]	equilibrium or thermometer temperature
T_s, [θ_s]	temperature constant in Sutherland's formula
U, V, [u,v]	velocity components
x, y, [ξ,η]	field coordinates
Y	Howarth's transversal coordinate (ft)
α	Blasius' constant, 1.328
β_m, β_{loc}	heat transfer coefficient, mean and local (ft-lb/ft^2 sec deg)
γ	c_p/c_v
δ	boundary layer thickness
δ_1	displacement thickness
ζ	solution of Blasius' equation [used in a different sense in equation (4.10)]
η	Blasius' variable $\frac{1}{2}y(U_\infty/x\nu_\infty)^{1/2} = \frac{1}{2}y(\text{Re}/xL)^{1/2}$ (This letter is used in a different sense in section IV, 2A, 3B, 3C.)
θ	momentum thickness (also identical with $t = T/T_\infty$ in section IV, 3C and end of section V, 3).
λ	form parameter in Pohlhausen's quartic approx. method
μ, [$\bar{\mu}$]	coefficient of viscosity (sl/ft sec)
ν, [$\bar{\nu}$]	μ/ρ, kinematic viscosity (ft^2/sec)
ρ, [$\bar{\rho}$]	density
σ	$c_p\mu/k$, Prandtl's number
τ, [c_τ]	viscous stress $\mu\partial U/\partial y$ (c_{τ_0} is the local wall stress coefficient)
$\chi(x,y)$	dependent variable in Howarth's boundary layer equation, equation (4.49)
Ψ, [ψ]	stream function
ω	exponent in the viscosity power law $\mu \propto T^\omega$

Subscripts

w	value of a boundary layer function for $y = 0$ (may still depend on x)
∞	value of a boundary layer function for $y = \infty$ (may still depend on x)
s	standard state to be specified (also referring to Sutherland's law)
S	specifying stagnation values at the boundary layer edge

I. Introduction

The problem of the mechanical and thermodynamical interaction between a stream of gas at high speed and a rigid surface essentially parallel to it has received, within the last ten years, a great deal of attention since it is of fundamental importance for high-speed aeronautical engineering as well as for the design and construction of gas turbines. It is true that a satisfactory theory of the interaction should include both laminar and turbulent flow in the boundary layer, but the dynamics and thermodynamics of the boundary layer must be cleared first in the laminar case before the complete problem can be attacked.

This report presents a survey of methods and results concerning *laminar* boundary layers. Only the two-dimensional (plane) problem is discussed; two-dimensional solutions can often be applied to axisymmetric problems without undue loss of accuracy. A remark about the general equivalence of the plane and axisymmetric problems is found in section V, 4.

Non-steady flow has so far mainly been considered in connection with questions of boundary layer stability, and recent reviews of this topic are available;* the present report can thus be restricted to *steady* b.l. flow.

Although the mathematics used in the boundary layer literature presents no particular conceptual difficulties, it is complicated on account of the number of parameters involved and because of the different approaches tried by different authors. The study of the original literature thus requires a patient reader. Under these circumstances it seemed desirable to put together what may be called a guide to rather than a review of the existing literature on the subject. The core of this report is therefore an organized representation of the various formal approaches that have evolved; it is written with the intent to show the common traits.

A never gratifying task is the choice of a suitable set of notations for a paper like this. The aim, of course, should be to retain as many of the original symbols as possible without sacrificing the readability of the review. It is hoped that the notation here chosen represents a reasonable compromise.

II. The Basic Equations

1. The Equations of the Viscous Compressible Fluid

To establish the equations of boundary layer theory, we have to start from the equations of motion for the *steady* flow of a *compressible*

* Chiarulli, P., and Freeman, J. C., Tech. Rept. No. F-TR-1197-1A (1948), Air Mat. Com., Wright Field, Ohio.

Lester Lees, N.A.C.A. Tech. Rept. No. 876 (1947).

fluid with *variable* viscosity. Assuming a *two-dimensional* field of flow and disregarding external forces such as gravity, the equation of motion for a fluid particle of unit volume is

$$\rho(\mathbf{q}\ \mathrm{grad})\ \mathbf{q} = \text{sum of surface reactions} \tag{2.1}$$

where $\mathbf{q}$ is the vector of the particle velocity with components U and V, and the symbol ($\mathbf{q}$ grad) stands for $U\partial/\partial x + V\partial/\partial y$. In the case of invariable density ρ and viscosity μ equation (2.1) reduces to the well-known Navier-Stokes equations, the surface reactions being given by $-\mathrm{grad}\ p + \mu\nabla^2\mathbf{q}$. In general, however, the reactive force equals the divergence of the stress tensor $-(pI + P)$, I being the unit tensor. The components $-p_{ik}$ of $-P$ are related to the rate of strain tensor $\dot\epsilon_{ik}$ and to its first scalar $\dot\theta = \dot\epsilon_{11} + \dot\epsilon_{22}$ by

$$-p_{ii} = 2\mu\dot\epsilon_{ii} + \lambda\dot\theta, \quad -p_{ik} = 2\mu\dot\epsilon_{ik} \tag{2.2}$$

According to a widely accepted simplification which is certainly legitimate for monatomic gases, the "second" coefficient of viscosity is taken as

$$\lambda = -\frac{2\mu}{3} \tag{2.2a}$$

Since the strain rates $\dot\epsilon$ are connected with the derivatives of the particle velocity by

$$\begin{aligned} \dot\epsilon_{11} &= U_x, \quad 2\dot\epsilon_{12} = U_y + V_x \\ 2\dot\epsilon_{21} &= V_x + U_y, \quad \dot\epsilon_{22} = V_y \end{aligned} \tag{2.2b}$$

equations (2.2), (2.2a), and (2.2b) permit one to express the reactive force, $-\mathrm{div}\ (pI + P)$, in terms of the spatial derivatives[1] of U, V, and μ. Equations (2.1) then assume the following form:

$$\begin{aligned} \rho UU_x + \rho VU_y = -p_x + \mu(U_{xx} + U_{yy}) + \frac{\mu}{3}(U_{xx} + V_{yx}) \\ + 2\mu_x U_x + \mu_y(U_y + V_x) - \tfrac{2}{3}\mu_x(U_x + V_y) \end{aligned} \tag{2.3}$$

$$\begin{aligned} \rho UV_x + \rho VV_y = -p_y + \mu(V_{xx} + V_{yy}) + \frac{\mu}{3}(U_{xy} + V_{yy}) \\ + 2\mu_y V_y + \mu_x(V_x + U_y) - \tfrac{2}{3}\mu_y(U_x + V_y) \end{aligned} \tag{2.4}$$

These equations together with the continuity equation

$$\frac{\partial}{\partial x}(\rho U) + \frac{\partial}{\partial y}(\rho V) = 0 \tag{2.5}$$

[1] The influence of higher derivatives has been considered by G. Viguier, Les équations de la couche limite dans le cas de gradients de vitesse élevés, *Compt. rend.*, 224, 715 (1947).

which expresses the conservation of mass of the fluid particle in steady flow, constitute the *equations of motion* of the fluid.

When the temperature T is included among the variables of the problem, we can formulate two further relations, viz., the *equation of state* and the *viscosity law* of the fluid. Throughout this paper we assume the validity of the perfect gas law

$$p = \rho RT \tag{2.6}$$

and consider the viscosity as a function of the temperature alone:

$$\mu = \mu(T) \tag{2.7}$$

Finally we use the *energy equation* to express that the flow is *adiabatic*,* i.e., no heat is introduced into an element except from the surrounding particles. Then the equation states that the rate of change of the total kinetic energy (= intrinsic plus kinetic energy) contained in a space-fixed volume element equals the rate at which work is done by the external forces and surface reactions, plus the rate at which total kinetic energy and heat is conducted into the element. In steady flow this energy balance can be expressed by the aid of (2.3), (2.4), (2.5) [see (1), p. 603ff.] in the form

$$\rho \mathrm{q} \cdot \operatorname{grad} E_i + p \operatorname{div} \mathrm{q} = \phi + \operatorname{div} (k \operatorname{grad} T) \tag{2.8}$$

where E_i is the intrinsic energy per unit of mass and k is the thermal conductivity, a function of T alone; the dissipation function ϕ is given by

$$\phi = \mu[2(U_x^2 + V_y^2) + (V_x + U_y)^2 - \tfrac{2}{3}(U_x + V_y)^2] \tag{2.8a}$$

Equation (2.8) is valid for any fluid whose intrinsic energy is a function of T alone [cf. (18), p. 4].

We now assume in addition to (2.6) that $E_i = c_v T$ with constant c_v and introduce the enthalpy per unit of mass

$$I = c_v T + \frac{p}{\rho} = c_p T \tag{2.9}$$

Using (2.5) one has

$$p \operatorname{div} \mathrm{q} + \mathrm{q} \cdot \operatorname{grad} p = \operatorname{div} \left[\left(\frac{p}{\rho} \right) (\rho \mathrm{q}) \right] = \rho \mathrm{q} \cdot \operatorname{grad} \frac{p}{\rho}$$

hence (2.8) may be written in the alternative form

$$\rho \mathrm{q} \cdot \operatorname{grad} I - \mathrm{q} \cdot \operatorname{grad} p = \phi + \operatorname{div} (k \operatorname{grad} T) \tag{2.10}$$

* In the wider sense of the word.

2. The Equations of the Boundary Layer

The limiting process that leads to the conventional boundary layer equations in the incompressible case (2,3) can also be applied in the present case. Let us first introduce dimensionless variables

$$u,\ v,\ \xi,\ \eta,\ \bar{\rho},\ \bar{p},\ t,\ \bar{\mu}$$

by reference to a standard state ρ_0, p_0, $T_0 = p_0/(R\rho_0)$, and to corresponding standard values $\mu_0 = \mu(T_0)$ and $k_0 = k(T_0)$. We further need a standard velocity $c = \sqrt{p_0/\rho_0}$, say, and a standard length l. Substituting

$$\begin{aligned} U &= cu,\ x = l\xi, \quad & T &= T_0 t, \quad & \rho &= \rho_0\bar{\rho}, \quad k = k_0\bar{k} \\ V &= cv,\ y = l\eta, \quad & p &= p_0\bar{p}, \quad & \mu &= \mu_0\bar{\mu} \end{aligned}$$

in equations (2.3), (2.4), (2.5), (2.8), one immediately verifies that the dimensionless form of the continuity equation is the same as in the original variables. When the transformed equations (2.3) and (2.4) are written in such a way that acceleration and pressure terms have their original form, the remaining terms exhibit the common factor $(\mu_0/\rho_0)/(cl) = \mathrm{Re}^{-1}$. The transformed energy equation (2.10) reads, if use is made of $R/c_p = (\gamma - 1)/\gamma$.

$$\bar{\rho}(ut_\xi + vt_\eta) = \frac{\gamma - 1}{\gamma}\,[(u\bar{p}_\xi + v\bar{p}_\eta) + \mathrm{Re}^{-1}\,\bar{\phi}] + \mathrm{Re}^{-1}\left[\frac{\partial}{\partial\xi}\,(\sigma^{-1}\bar{k}t_\xi) + \frac{\partial}{\partial\eta}\,(\sigma^{-1}\bar{k}t_\eta)\right] \qquad (2.11)$$

Here $\bar{\phi} = \bar{\mu}[2(u_\xi{}^2 + v_\eta{}^2) + \cdots]$ as in (2.8a) and

$$\sigma = \frac{c_p\mu_0}{k_0} = \frac{c_p\mu}{k}$$

is Prandtl's number which from now on we assume to be a characteristic physical constant of the fluid, but entirely independent of its state.[2]

We now apply our equations to a fluid of small viscosity, that is, we want to retain only those terms that are significant at large Reynolds numbers. Simply to let $\mathrm{Re} \to \infty$ would altogether throw out the influence of viscosity. However, a consideration of the actual flow of a viscous fluid past a straight boundary ($\eta = 0$) shows the η-component of grad μ to be so large that the transition from $\mathrm{q} = 0$ at the boundary to the value of q in the free stream is essentially completed within a thin

[2] Elementary kinetic theory gives $\sigma = 4\gamma/(9\gamma - 5)$ (Eucken, 1913). For criticism of this result see Chapman and Cowling, The Mathematical Theory of Non-Uniform Gases, Cambridge, 1939. Actual measurement gives 0.715 for air at ordinary temperature and a few percent more for very low temperature.

layer. This behavior of the flow, which similarly occurs along any smooth boundary for not too small Re-number, suggests to go to the limit $\mathrm{Re} \to \infty$ under simultaneous preservation of the weight of the term $\partial u/\partial\eta$. Trying to do so, one may put

$$\frac{\partial}{\partial\eta} = \mathrm{Re}^n \frac{\partial}{\partial\eta'}$$

with positive n and stipulate that $\partial u/\partial\eta'$ *tend to a non-vanishing limit when* $\mathrm{Re} \to \infty$. Then the terms $\bar\mu u_{\eta\eta} + \bar\mu_\eta u_\eta$ in the dimensionless form of equation (2.3) transform into $\mathrm{Re}^{2n}(\bar\mu u_{\eta'\eta'} + \bar\mu_{\eta'} u_{\eta'})$, hence the factor Re^{-1} is canceled if one makes $n = \frac{1}{2}$. On the other hand, the term $\bar\rho v u_\eta$ of the left member of the same equation would transform into $\mathrm{Re}^{1/2}\bar\rho v u_{\eta'}$. It therefore becomes necessary to make the additional provision that $v\mathrm{Re}^{1/2}$ remains finite with ever increasing Re. One *assumes* then

$$u_{\eta'} = O(\mathrm{Re})^{1/2}, \quad v = O(\mathrm{Re}^{-1/2})$$

This calls for the following affine transformation to be applied to a narrow strip adjacent to the wall:

$$\xi = \xi', \qquad \eta = \eta' \,\mathrm{Re}^{-1/2} \tag{2.12}$$

it assigns to any layer of finite breadth η an infinite breadth in the coordinate η' when $\mathrm{Re} \to \infty$. At the same time, the velocity components u, v must be subjected to

$$u = u', \qquad v = v' \,\mathrm{Re}^{-1/2}; \tag{2.12a}$$

it is, of course assumed that σ remains the same in the sequence of flows that belong to the ever increasing sequence of Re-values.

In writing the resulting system of equations we omit the primes on ξ', η', u', v'. Then the equations of motion reduce to

$$\bar\rho(uu_\xi + vu_\eta) = -\bar p_\xi + \frac{\partial}{\partial\eta}\left(\bar\mu \frac{\partial u}{\partial\eta}\right) \tag{2.13}$$

$$0 = -\bar p_\eta \tag{2.14}$$

the continuity equation retains its form

$$\frac{\partial}{\partial\xi}(\bar\rho u) + \frac{\partial}{\partial\eta}(\bar\rho v) = 0 \tag{2.15}$$

and the energy equation reads, when (2.14) is taken into account,

$$\bar\rho(ut_\xi + vt_\eta) = \frac{\gamma - 1}{\gamma}\,[u\bar p_\xi + \bar\mu u_\eta^2] + \sigma^{-1}\frac{\partial}{\partial\eta}\left(\bar k \frac{\partial t}{\partial\eta}\right) \tag{2.16}$$

It is a simple matter to rewrite these equations in the dimensional variables, but the transformation formulas contain the Reynolds number ($\eta = \mathrm{Re}^{1/2}\, y/l$ and $v = \mathrm{Re}^{1/2}\, V/c$), hence the physical meaning of the system thus obtained [see the following equations (2.17)–(2.20)] is fundamentally different from that of the original system (2.3), (2.4), (2.5), (2.8). In the retransformation we naturally have to use a large but *finite* Re, hence the dimensional equations describe *approximately* the flow in the immediate neighborhood of the wall for a certain range of *large* Re-numbers (but not so large as to make the flow *turbulent*). Since the transition from (2.3), etc., to (2.13), etc., assigns to an arbitrarily small $y > 0$ an infinite η, the free stream boundary is at infinity in the coordinate η and remains there in the coordinate y of equations (2.17), etc., the divisor $\mathrm{Re}^{1/2}/l$ now being *finite:* In other words, the boundary layer fills the whole range $0 < y < \infty$. Hence it is only possible to introduce the concept of boundary layer thickness, when solutions of the system (2.17)–(2.20) and their asymptotic behavior for large y-values have become known [cf. (28), pp. 13, 17].

It makes no difficulty to carry out the same limiting process in the case of a smoothly curved cylindrical wall. For constant ρ and μ this has been done in reference 3, p. 517 [see also (1), p. 121] and the same method works in the present case. The resulting system of equations is identical with the one for the plane wall, the meaning of x and y in the general case being the arc length along the wall and the distance along the normal to the wall respectively.[3] The dimensional boundary layer equations follow here:

$$\rho(UU_x + VU_y) = -p_x + \frac{\partial}{\partial y}\left(\mu \frac{\partial U}{\partial y}\right) \tag{2.17}$$

$$p_y = 0 \tag{2.18}$$

$$\frac{\partial}{\partial x}(\rho U) + \frac{\partial}{\partial y}(\rho V) = 0 \tag{2.19}$$

$$\rho c_p(UT_x + VT_y) - p_x U = \mu U_y^2 + \frac{\partial}{\partial y}\left(k \frac{\partial T}{\partial y}\right). \tag{2.20a}$$

The alternative form of (2.20a), developed from (2.8), is

$$\rho c_v(UT_x + VT_y) + p(U_x + V_y) = \mu U_y^2 + \frac{\partial}{\partial y}\left(k \frac{\partial T}{\partial y}\right) \tag{2.20b}$$

[3] According to this derivation the second equation of motion for a curved wall is $p_y = 0$ and not $p_y = \rho \kappa u^2$, κ being the curvature. [See also (4, 28) and on the other hand (1), p. 120 and (18), p. 6.] Thus it is unnecessary to distinguish between a plane and curved cylindrical wall, p_x being the only "external" parameter of the problem.

The usual boundary conditions for viscous flow require $\mathfrak{q} = 0$ at the wall, while the wall temperature may be prescribed as a function of x:

$$y = 0: \qquad U = V = 0, \quad T = T_w(x) \tag{2.21}$$

(Solutions that would satisfy a given T_y-distribution have not become known so far, except in the case of a thermally insulated wall where $T_y = 0$.)

At the free stream boundary the T-distribution may be prescribed, but only the x-component of $\mathfrak{q}$ may be given in advance:

$$y = \infty: \qquad U_\infty = U_\infty(x), \quad T_\infty = T_\infty(x) \tag{2.21a}$$

In the actual application of (2.21a) two further restrictions are made:

(*a*) one identifies U_∞ with the tangential velocity of a certain inviscid flow past the given rigid boundary, disregarding the contribution of V to the velocity $\mathfrak{c}_\infty$.

(*b*) the flow at the boundary of the free stream is assumed isentropic [or piecewise isentropic (shock)]; then only one of the three functions T_∞, U_∞, p_∞ may be chosen freely. Once this choice has been made, p, p_x *and* the conditions (2.21a) are fixed. In other words, the "external" parameter p_x and the functions $U_\infty(x)$ and $T_\infty(x)$ must belong to the same stream filament of a certain isentropic flow.

III. The Simple Problem

This section gives a brief survey of the older papers that deal with the "Crocco integral" of the plate problem; thermodynamic and analytic aspects will be found in sections III, 2 and IV. The term "simple" is meant to specify: $\sigma = 1$; $p_x = 0$, which implies $T_\infty =$ constant, $U_\infty =$ constant; $T_w \neq$ constant, including the case $(T_y)_w = 0$.

1. *History and General Character of the Solution*

The uniform flow along a *thermally insulated* plane wall was studied in terms of boundary layer theory by Pohlhausen in 1921 [plate thermometer problem (5)] for small flow velocity and small temperature differences, hence constant ρ and μ. With these simplifications the triplet (2.17)–(2.19) may be solved independently from (2.20); Pohlhausen can therefore use the known solution of Blasius' plate problem[4] (9).

[4] For constant ρ, μ, and $p_x = 0$, (2.17)–(2.19) can be reduced to $\zeta''' + \zeta\zeta'' = 0$, with $\zeta = \zeta' = 0$ for $\eta = 0$, and $\zeta' = 2$ for $\eta = \infty$, where $\zeta' = 2u = 2U/U_\infty$ and $\eta = \frac{1}{2}y[U_\infty/(x\nu)]^{1/2}$. The function $\zeta(\eta)$ that solves this boundary value problem is known as Blasius' function. (Note that in reference 9 Blasius writes ξ instead of η.) The number $\alpha = \zeta''(0) = 2u_w'$ will be called Blasius' constant; η is often designated as Blasius' variable.

He determines from (2.20) the excess of the equilibrium temperature T_e, shown by the thermometer, over the actual stream temperature T_∞ as a function of σ and finds for $\sigma = 1$ the result $T_e = T_\infty + U_\infty^2/2c_p$. Thus the wall temperature $T_w = T_e$ equals the free stream stagnation temperature or, stated differently, the point function $c_pT + U^2/2$ assumes the same value at $y = 0$ and $y = \infty$.

Ten years later Busemann in his report on gas dynamics (6) was able to point out that this function[5] (also written $I + U^2/2$) is actually *constant throughout the entire boundary layer* along the plate and obviously so in the uniform free stream; and that this is also true for a *compressible* fluid with T-dependent μ, provided $\sigma = 1$. In other words, $c_pT + U^2/2 =$ constant is a particular integral of the boundary layer equations with $\sigma = 1$ and $p_x = 0$, fulfilling the boundary condition $T_y = 0$ for $y = 0$; the constant must equal the stagnation enthalpy of the free stream.

In the following year (1932) Crocco (7) discovered the more general integral

$$c_pT + \frac{U^2}{2} = c_p(aU + b) \tag{3.1}$$

which solves the same problem for positive or negative heat exchange between fluid and plate when constant $T_w \neq T_e$ and T_∞ is prescribed.

Busemann (8) was the first to attempt the determination of the flow field $U(x,y)$ that belongs to the Crocco integral and is given by equations (2.17)–(2.19) with $p_x = 0$. By introduction of the variable $y/\sqrt{x}$ the problem is reduced to a boundary value problem of an ordinary third order differential equation for $u = U/U_\infty$. This equation includes as a special case the Blasius equation, the numerical solution of which is very accurately known, but this circumstance is not fully exploited. The velocity profile u vs. $\eta = \frac{1}{2}y[U_\infty/(x\nu_\infty)]^{1/2}$ is computed under the assumption $\mu \propto T^{0.5}$ for the case of no heat transfer and shown diagrammatically at a free stream Mach number $M = 8.8$. Over the whole range $0 < M < \infty$ the initial slope of the velocity profiles steepens only by 20% of its value at $M = 0$, where it coincides with half the value of Blasius' constant

$$\alpha = 1.328$$

the profiles themselves straighten out with increasing M.

On the basis of this last result Kármán (10) [see also (11)] assumed a *linear* velocity profile and estimated the drag coefficient[6] C_{D_f} by means of

[5] Often called total energy or total heat, per unit of mass.

[6] Coefficient of skin friction referring to the wetted surface.

the momentum theorem. For large M_∞ he obtains

$$C_{D_f}\sqrt{\mathrm{Re}} \sim 1.57\left(1+\frac{\gamma-1}{2}M^2\right)^{-(\omega-1)/2} \tag{3.2}$$

where ω is the exponent in the viscosity law $\mu \propto T^\omega$ and Re refers to the length of the wetted area. Note the great practical importance of the correct choice of ω.

In 1939, Kármán and Tsien (11) reexamined the question of the velocity field by means of a more efficient mathematical method. They subject equation (2.17) to the Mises transformation (2,3) which consists in the introduction of the stream function ψ as independent variable instead of y. The resulting boundary value problem involves again a partial differential equation for $u = U/U_\infty$ in the independent variables x and Ψ. In analogy to the variable $y/\sqrt{x}$, they introduce the independent variable $\psi/\sqrt{x}$ and thus obtain a boundary value problem for u that involves an ordinary differential equation. This problem has a unique solution which therefore must be equivalent to the solution of the original problem.[7] The parameters ρ and μ that appear in the equation are functions of T and, by (3.1), functions of u. An essential point is, however, that the equation can be solved by successive approximation, the first approximation being the known solution of the Blasius equation.

The numerical computations are carried out on the basis of the viscosity law $\mu \propto T^{0.76}$, which is a good approximation to reality at ordinary temperatures, first for the boundary condition $(T_y)_w = 0$ (Fig. 1) and, secondly, for $T_\infty = 4T_w$ (Fig. 2). The diagrams show the velocity and temperature fields vs. 2η in the two cases for Mach numbers up to 10. In the opinion of the authors the very large temperature

[7] The fact that U depends only on $y/\sqrt{x}$ is directly related to the transformation that connects the system (2.13), etc., with the system (2.17) etc. A completely specified boundary layer flow past a straight wall with *constant* boundary values T_w, T_∞, U_∞ can be transformed in many ways to nondimensional quantities. We may, for example, choose a set of different characteristic lengths l_i but keep the other scale standards such as T_0, c, etc., constant, obtaining in this way a certain set of transformations. But l does not occur explicitly in the system (2.13), etc. *nor does it appear in the boundary conditions of the problem*, hence there is only one dimensionless system with one dimensionless solution. This solution assigns to the point $\xi = 1$, η, let us say, the ξ-velocity u. In the physical plane the velocity $U = cu$ is then found at all points $x_i = l_i$, $y_i = l_i\eta/\mathrm{Re}_i^{1/2} \propto \eta\sqrt{l_i}$. Since the l_i are arbitrary, it follows that U must be the same whenever $y_i/\sqrt{x_i}$ is the same, which implies that U depends on x and y only through $y/\sqrt{x}$. The same conclusion applies to T, $\rho(T)$, $\mu(T)$ and $V\sqrt{x}$.

The equivalence of the substitutions $y/\sqrt{x}$ and $\Psi/\sqrt{x}$ becomes apparent, if one considers that in $\Psi = \int(\Psi_x dx + \Psi_y dy)$ the first and second integrands have the forms $[f_1(y/\sqrt{x})]/\sqrt{x}$ and $f_2(y/\sqrt{x})$ respectively [cf. equations 4.1]; one finds then easily that $\Psi/\sqrt{x}$ is a function of $y/\sqrt{x}$.

rise at the insulated wall for higher M calls for a revision of the assumed μ,T-relation and for consideration of the radiation losses.

In the subsequent period the plate problem has been treated under more realistic assumptions about σ ($\sigma_{air} = 0.72$) and the μ,T-relation. We think here of the later papers of Crocco, of the work of Hantzsche and Wendt, and Emmons and Brainerd. Also Howarth, in his analysis of the general boundary layer problem on the basis of $\sigma = 1$, $\mu \propto T$, returns to the simple problem as an example for his approach (17).

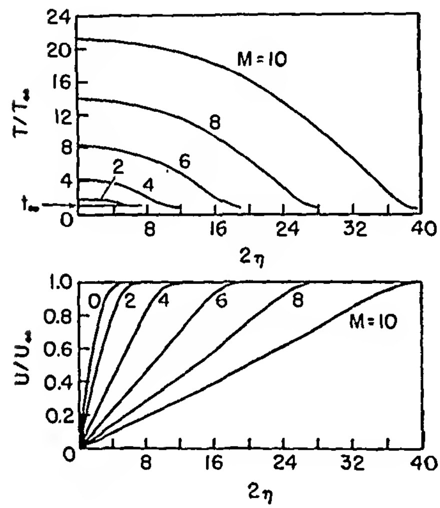


FIG. 1

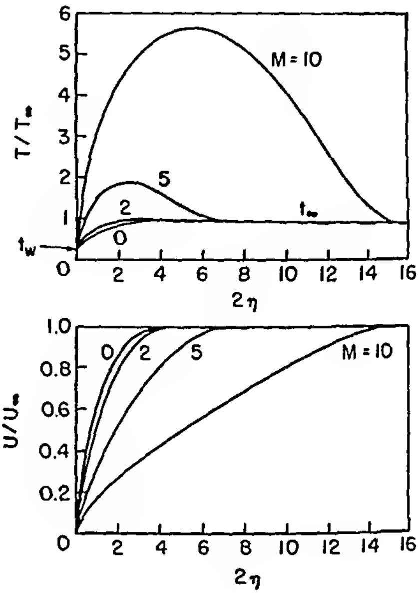


FIG. 2

2. Heat Transfer Characteristics

Let us first rewrite the energy equation (2.20a) so as to make the existence of the Crocco integral (3.1) evident. Since the latter involves the total energy $I + U^2/2$, we shall introduce the function

$$T^* = T + \frac{U^2}{2c_p} \tag{3.3}$$

instead of T. (The term "stagnation temperature" will be applied to T^* although the flow *within* the boundary layer is, of course, not isentropic; T^* is the temperature of a fluid element on being *stopped isentropically.*) Divide (2.20a) by c_p and rewrite kT_y/c_p as follows:

$$\frac{k}{c_p}\,T_y = \mu\left[\left(\frac{1}{\sigma} - 1\right) + 1\right]T_y = \mu\left(\frac{1}{\sigma} - 1\right)T_y + \mu\left(T_y{}^* - \frac{1}{c_p}\,UU_y\right)$$

Then (2.20a) becomes

$$\rho \mathbf{q} \cdot \text{grad } T = \frac{U}{c_p} p_x + \left(\frac{1}{\sigma} - 1\right) \frac{\partial}{\partial y} (\mu T_y) + \frac{\partial}{\partial y} (\mu T_y^*) - \frac{1}{c_p} \frac{\partial}{\partial y} (\mu U U_y) + \frac{\mu}{c_p} U_y^2 \tag{3.4}$$

On the other hand, multiply (2.17) by U/c_p and rearrange the second member. Then

$$\rho \mathbf{q} \cdot \text{grad} \left(\frac{U^2}{2c_p}\right) = -\frac{U}{c_p} p_x + \frac{1}{c_p} \frac{\partial}{\partial y} (\mu U U_y) - \frac{\mu}{c_p} U_y^2 \tag{3.5}$$

Equations (3.4) and (3.5) add up to

$$\rho \mathbf{q} \cdot \text{grad } T^* = \frac{\partial}{\partial y} (\mu T_y^*) + \left(\frac{1}{\sigma} - 1\right) \frac{\partial}{\partial y} (\mu T_y) \tag{3.6}$$

When $\sigma = 1$,

$$T^* = \text{constant} \tag{3.9'}$$

is thus always a particular integral *regardless of the distribution of* p (or T_∞, U_∞) at the free stream boundary; it solves the simple problem in the case of no heat transfer ($T_y = -UU_y/c_p$ vanishes at the wall) for any given distribution of U_∞, say, and therefore represents the *most general integral* (*in the sense of restriction* (*b*) at the end of section II, 2) *of the simple boundary layer problem, when* $T_y = 0$ *is prescribed at the wall.* Note that it implies $T_w =$ constant.

When, in addition, p is constant, (3.6) and (2.17) read

$$\rho \mathbf{q} \cdot \text{grad } T^* = \frac{\partial}{\partial y} (\mu T_y^*) \tag{3.7}$$

$$\rho \mathbf{q} \cdot \text{grad } U = \frac{\partial}{\partial y} (\mu U_y) \tag{3.8}$$

If we now search for a solution T^* that is a function of U alone, the substitution of such a $T^*(U)$ in (3.7) must not set a new condition for U. But that will always happen unless

$$T^* = aU + b \tag{3.9}$$

The new integral (3.9) belongs to the boundary condition $T_w =$ constant ($= b$), but applies only when p, and thus T_∞ and U_∞, are constant. *Hence (3.9) solves the simple boundary layer problem in the case of a uniform free stream along a boundary kept at constant temperature.*

Expressing (3.9) in terms of

$$u = \frac{U}{U_\infty}, \quad t = \frac{T}{T_\infty} \tag{3.10}$$

and introducing the free stream Mach number M by

$$\frac{U_\infty^2}{2c_pT_\infty} = \frac{\gamma - 1}{2} M^2 \tag{3.11}$$

one has first

$$t = t_w + au\frac{U_\infty}{T_\infty} - u^2\frac{\gamma - 1}{2} M^2 \tag{3.12}$$

For $u = 1$ one has $t = 1$, which determines a in terms of the other constants, hence $a = [(\gamma - 1)M^2/2 - (t_w - 1)]/(U_\infty/T_\infty)$, and

$$t = t_w + u\left[\frac{\gamma - 1}{2} M^2 - (t_w - 1)\right] - u^2\frac{\gamma - 1}{2} M^2 \tag{3.13}$$

The direction of the heat transfer depends on the sign of

$$\left(\frac{\partial t}{\partial y}\right)_w = \left(\frac{aU_\infty}{T_\infty}\right)\left(\frac{\partial u}{\partial y}\right)_w. \tag{3.13a}$$

Now, $(\partial u/\partial y)_w > 0$ for an "unseparated" boundary layer, whence

heat flows to the fluid if $\frac{\gamma - 1}{2} M^2 < t_w - 1$

heat flows to the wall if $\frac{\gamma - 1}{2} M^2 > t_w - 1$.

The first case is, of course, only possible if $T_w > T_\infty$, but this condition alone does not ensure cooling action of the stream on the wall. The second case occurs whenever $T_w < T_\infty$ and is of importance if heat is to be removed from a hot stream. Again, $T_w < T_\infty$ alone does not ensure cooling action of the wall on the stream, since the presence of the wall causes heat production in the boundary layer. It is true that heat is removed from the stream, but whether the stream is cooled or heated depends on the balance between the heat produced and the heat transferred to the wall which cannot be calculated without knowledge of the u-distribution [cf. the example in (11), p. 230].

The coefficient aU_∞/T_∞ in (3.12) is directly related to the ratio (heat transferred)/(wall friction). Let $-k_w(T_y)_w \equiv q_w$ be the heat flow to the fluid per unit of time and area, and τ_w the frictional stress at the wall, then there is always

$$\frac{q_w}{\tau_w} = -\frac{k_w}{\mu_w}\frac{(T_y)_w}{(U_y)_w} \tag{3.14$'$}$$

hence, whenever T is a function of U alone,

$$\frac{q_w}{\tau_w} = -\frac{k_w}{\mu_w}\left(\frac{dT}{dU}\right)_w \tag{3.14$''$}$$

independently of x. In the present case, (3.9) yields $(dT/dU)_w = a$ and

$$\frac{q_w}{\tau_w} = -\frac{k_w}{\mu_w}\frac{T_\infty}{U_\infty}\left[\frac{\gamma - 1}{2} M^2 - (t_w - 1)\right] \tag{3.14}$$

which represents Reynolds' analogy[5] (1874) corrected for compressibility, in the case $\sigma = 1$.

Further analysis of the heat exchange characteristics must be made in terms of a certain parameter K_0 that is associated with the velocity field $U(x,y)$. The function $U(x,y)$ is the (unique) solution of (3.8) and (2.19), when ρ and μ are known functions of U by (3.13); we shall have to use the fact that $U(x,y)$ is a function of $\Psi/\sqrt{x}$ (cf. p. 31, ftn.).

According to the momentum theorem [(1), p. 132] the stress at the station x of a plate of length L is given by

$$\tau_w = \frac{\partial}{\partial x}\int_0^\infty \rho(U_\infty - U)U\,dy = U_\infty^2 \frac{\partial}{\partial x}\int_0^\infty \rho(1 - u)u\,dy \tag{3.15}$$

and the drag coefficient is

$$C_{D_f} = 2\left(\int_0^L \tau_w dx\right)/\rho_\infty L U_\infty^2 \tag{3.16}$$

The integration with respect to x can be performed, if the particular form of the dependence of u on x and y is utilized: u is a function of the dimensionless $\zeta = \psi/\sqrt{x/L}$, or of

$$\zeta = \frac{\Psi}{U_\infty L}\sqrt{\mathrm{Re}}/\sqrt{x/L} \tag{3.17}$$

where $\mathrm{Re} = U_\infty L/\nu_\infty$. Since $\rho U = \rho_\infty \Psi_y$ [see (4.1)], and ρ is also a function of ζ, we may substitute for dy in (3.15)

$$dy = \rho_\infty \sqrt{\frac{xL}{\mathrm{Re}}}\frac{d\zeta}{\rho u} \tag{3.18}$$

On identifying the streamline $\Psi = 0$ with the plate surface and excluding the singular point $x = 0$, the range $0 < y < \infty$ is mapped on $0 < \zeta < \infty$. Substituting as indicated, we obtain

$$\tau_w = \rho_\infty U_\infty^2 \left(\frac{L}{\mathrm{Re}}\right)^{1/2} \frac{d\sqrt{x}}{dx}\int_0^\infty (1 - u)d\zeta \tag{3.19}$$

and (3.16) yields

$$C_{D_f}\sqrt{\mathrm{Re}} = 2\int_0^\infty (1 - u)d\zeta \;(= 2K_0) \tag{3.20}$$

[5] Papers on Mech. and Phys. Subjects, I, p. 81. This relation was rediscovered by Prandtl in 1910, *Physik. Z.*, 11, 1072.

This important relation is the root of the (approximate) equation (3.2). The integral in (3.20), which we denote by K_0, depends on the coefficients of the polynomial (3.13) and on the viscosity law.[9] Thus, for fixed thermal boundary conditions and a specified viscosity law, K_0 is still a function of M (see Fig. 2 in (11) and our Fig. 5). In the particular case $\rho\mu$ = const, that is, $\mu \propto T$, K_0 no longer depends on T_∞, T_w, M; its numerical value is then

$$K_0 = \frac{\alpha}{2} = 0.664$$

and this remains true even if $\sigma \neq 1$, as will be seen later [(16), p. 18]. When we consider K_0 as the basic physical parameter of the problem, then (3.19) describes the dependence of $\tau_w = \mu_w(\partial U/\partial y)_w$ on x and K_0, and the slope of the velocity profile can be expressed in the same terms:

$$\left(\frac{\partial U}{\partial y}\right)_w = \frac{U_\infty}{2L}\frac{\mu_\infty}{\mu_w}\left(\frac{\mathrm{Re}}{x/L}\right)^{1/2} K_0 \tag{3.21}$$

According to (3.14), q_w/τ_w is independent of x, hence

$$\frac{q_w}{\tau_w} = \frac{\int_0^L q_w dx}{\int_0^L \tau_w dx} \equiv \frac{Q_w}{F_w} \tag{3.22$'$}$$

Using (3.14″) and (3.16), we obtain for the heat transmitted to the fluid through the area $L \times 1$ per unit of time

$$Q_w = -\rho_\infty U_\infty{}^2 L \frac{k_w}{\mu_w}\left(\frac{\partial T}{\partial U}\right)_w \frac{K_0}{\sqrt{\mathrm{Re}}} \tag{3.22}$$

The mean heat transfer coefficient β_m may be defined as

$$\beta_m = (Q_w/L)/(T_w - T_\infty) \tag{3.23}$$

and the (dimensionless) Nusselt number

$$\mathrm{Nu} = \frac{Q_w}{k_\infty(T_w - T_\infty)} = \frac{\beta_m L}{k_\infty} \tag{3.24}$$

can be written in the form

$$\mathrm{Nu} = -\sqrt{\mathrm{Re}}\, K_0 \left(\frac{\partial t}{\partial u}\right)_w \Big/ (t_w - 1) = \sqrt{\mathrm{Re}}\, K_0 \left[1 - \frac{(\gamma - 1)M^2}{2(t_w - 1)}\right] \tag{3.25}$$

[9] The differential equation that determines $u(\zeta)$ contains the parameter $\bar{\rho}\bar{\mu}$. See later equation (4.10).

when the constancy of σ and c_p is used to eliminate k and μ. Introducing here the free stream stagnation temperature T_∞^* which, according to (3.3) and (3.11) equals $T_\infty + [(\gamma - 1)M^2/2]T_\infty$, we obtain

$$\mathrm{Nu} = \sqrt{\mathrm{Re}}\, K_0 \left[1 - \frac{T_\infty^* - T_\infty}{T_w - T_\infty} \right] \tag{3.25a}$$

Observing that K_0 is a slowly varying function of M we notice that the influence of compressibility on the heat transfer characteristics is mainly due to the increase of the excess stagnation temperature $T_\infty^* - T_\infty$ with increasing U_∞, when T_w and T_∞ are kept constant. Equation (3.22) may now be written

$$Q_w = -k_\infty \sqrt{\mathrm{Re}}\, K_0(T_\infty^* - T_w) \tag{3.25b}$$

This relation is due to Crocco (1932); the same law governs the heat transfer at low and high speed, but at low speed T_∞^* may be replaced by T_∞. The quantity T_∞^* is sometimes called the effective temperature of the gas stream. Also equilibrium or thermometer temperature are in use. All this is only correct if $\sigma = 1$.

IV. The Mathematics of Boundary Layer Theory

In this section we shall present a number of transformations that have evolved in the search for practical solutions of the boundary layer problem. The final aim in every case is, of course, the numerical tabulation of solutions, but let us try to distinguish for the purpose of the discussion between (1) transformations that work under suitable simplifying assumptions about the parameters μ, ρ, and σ, (2) transformations that adapt the compressible boundary layer equations to the application of approximate methods developed for the incompressible boundary layer, and, finally, (3) transformations that prepare the equations for a numerical computer (case $p_x = 0$ in (14,15), case p_x = constant in (18) which gives an excellent survey of the numerical-analytical methods available for this problem).

1. Transformations That Work under Suitable Simplifying Assumptions about the Parameters μ, ρ, and σ

This type includes (*a*) the Mises transformation applied by Kármán and Tsien (11) and quite recently by Chapman and Rubesin (19,21); (*b*) the transformation used by Crocco (12,16), and by Hantzsche and Wendt (13); (*c*) Howarth's transformation (17).

Kalikhman's transformation (20) prepares the boundary layer equations for the application of Pohlhausen's method. Howarth uses his

form of the boundary layer equations for the same purpose and also for an extension of the power series methods to the compressible case.

We start with a brief discussion of the transformations (*a*), (*b*), (*c*) whose common aim is separation of the unknowns, which is possible under certain conditions.

A. The Independent Variable Ψ *(von Kármán and Tsien, Chapman and Rubesin)—The Viscosity Law.* On introducing the stream function for compressible flow Ψ the continuity equation (2.19) is identically fulfilled:

$$U = \frac{\rho_\infty}{\rho} \Psi_y, \qquad V = -\frac{\rho_\infty}{\rho} \Psi_x \tag{4.1}$$

Here ρ_∞ is supposed to be constant; we thus assume a uniform free stream, i.e.,

$$p_x = 0 \tag{4.1a}$$

The stream function Ψ is now introduced as a new *independent* variable. The derivatives f_x and f_y of some $f(x,y)$ in terms of the new variables

$$S = x, \qquad \Psi = \Psi(x,y) \tag{4.2}$$

are

$$f_x = S_x \frac{\partial f}{\partial S} + \Psi_x \frac{\partial f}{\partial \Psi}, \qquad f_y = S_y \frac{\partial f}{\partial S} + \Psi_y \frac{\partial f}{\partial \Psi} \tag{4.3}$$

where the derivatives with respect to S and Ψ have been written in the ∂-notation. By (4.1) and (4.2) we obtain

$$f_x = \frac{\partial f}{\partial x} - \frac{\rho}{\rho_\infty} V \frac{\partial f}{\partial \Psi}, \qquad f_y = \frac{\rho}{\rho_\infty} U \frac{\partial f}{\partial \Psi} \tag{4.4}$$

when the dummy variable S is discarded. Hence

$$Uf_x + Vf_y = U \frac{\partial f}{\partial x} \tag{4.5}$$

and equations (2.17) and (2.20a) read after division by ρU, when k/c_p is replaced by μ/σ,

$$\frac{\partial U}{\partial x} = \frac{1}{\rho_\infty{}^2} \frac{\partial}{\partial \Psi}\left(\mu\rho U \frac{\partial U}{\partial \Psi}\right) \tag{4.6}$$

$$\frac{\partial T}{\partial x} = \frac{\mu}{\rho_\infty c_p} \frac{\rho}{\rho_\infty} U \left(\frac{\partial U}{\partial \Psi}\right)^2 + \frac{1}{\sigma\rho_\infty{}^2} \frac{\partial}{\partial \Psi}\left(\mu\rho\, U \frac{\partial T}{\partial \Psi}\right) \tag{4.7}$$

We add here for later reference the equation that is obtained when Ψ is introduced as a new *unknown* instead of U in equation (2.17), that being the original Prandtl-Blasius method of combining (2.17) and (2.19)

into one equation. Retaining the term p_x, we have to use in (4.1) some standard ρ_s instead of ρ_∞ which is now variable. The resulting equation is

$$\Psi_y\Psi_{xy} - \Psi_x\Psi_{yy} = -\frac{p_x}{\rho_s} + \frac{\partial}{\partial y}\left[\mu \frac{\partial}{\partial y}\left(\frac{1}{\rho}\Psi_y\right)\right] \tag{4.6a}$$

This is a *third order* equation, the form of the "viscous" term is such as to require further transformation, as in (4.6).

Returning to the Mises transformation, we introduce dimensionless variables in (4.6) and (4.7) as on p. 26 and specify the standard state by ρ_∞ and T_∞, set $c = U_\infty$ and l = plate length L, and take a dimensionless stream function

$$\psi = \frac{\Psi}{U_\infty L}\left(\frac{\text{Re}}{C}\right)^{\frac{1}{2}} = \frac{\Psi}{\sqrt{\nu_\infty U_\infty LC}} \tag{4.8}$$

where C is a dimensionless quantity that will be adjusted later. The system (4.6), (4.7) reads then

$$\frac{\partial u}{\partial \xi} = \frac{\partial}{\partial \psi}\left(\frac{\bar{\mu}\bar{\rho}}{C}u\frac{\partial u}{\partial \psi}\right) \tag{4.9a}$$

$$\frac{\partial t}{\partial \xi} = \frac{\bar{\mu}\bar{\rho}}{C}(\gamma - 1)M_\infty^2 u\left(\frac{\partial u}{\partial \psi}\right)^2 + \frac{1}{\sigma}\frac{\partial}{\partial \psi}\left(\frac{\bar{\mu}\bar{\rho}}{C}u\frac{\partial t}{\partial \psi}\right) \tag{4.9b}$$

Equations (4.9) and (4.10) become independent of each other in either of the two cases

$$(\alpha) \quad \sigma = 1$$
$$(\beta) \quad \bar{\mu}\bar{\rho} = C$$

For $\sigma = 1$, the analysis of our system has already been indicated in terms of the equivalent equations (3.7) and (3.8), and the particular integral (3.13), where t = second order polynomial in u, is available for the boundary condition T_w = constant. In that case equation (4.9b) need no longer concern us, and $\bar{\mu}$, $\bar{\rho}$ in (4.9a) are known functions of u. Equation (4.9a) with $C = 1$, together with substitution (3.17), which now reads $\zeta = \psi/\sqrt{\xi}$, leads directly to

$$-\frac{\zeta}{2}\frac{du}{d\zeta} = \frac{d}{d\zeta}\left(u\bar{\rho}\bar{\mu}\frac{du}{d\zeta}\right) \tag{4.10}$$

This equation marks the starting point of the computations in (11). [For an account of the iteration method which is used see (21) where the same problem is solved, under different boundary conditions.]

The assumption β

$$\bar{\mu}\bar{\rho} = C \tag{4.11}$$

especially with $C = 1$, has often been used. Howarth applied it together with $\sigma = 1$ in 1948 to obtain solutions for nonvanishing pressure gradient; *his method of attack will be presented later.* *For our present* purpose the physical significance of this and other assumptions about the viscosity law should be pointed out.

Because of (2.6) and (4.1a), (4.11) implies

$$\frac{\mu}{\mu_\infty} = \frac{CT}{T_\infty} \tag{4.11a}$$

and is therefore a special case ($\omega = 1$) of the power law $\mu \propto T^\omega$. The correct[10] physical law that represents accurately the actual variation of μ for a gas above the critical temperature is Sutherland's formula

$$\mu \propto \frac{T^{3/2}}{T + T_s} \tag{4.12}$$

where the value of the constant T_s is about 120°C (216°F) for air. If (4.11) is to be approximated by a power law with a simple rational exponent then, according to (18), p. 15, one should use

$\frac{8}{9}$ for $90° < T < 300°$K
$\frac{3}{4}$ for $250° < T < 600°$K

but the straight approximation $\mu/\mu_\infty = T/T_\infty$ that meets

$$\frac{\mu}{\mu_\infty} = \left(\frac{T}{T_\infty}\right)^{3/2} \frac{T_\infty + T_s}{T + T_s} \tag{4.12a}$$

at $T = T_\infty$ [it is tangential to (4.12a) when $T_\infty = T_s$] becomes unsatisfactory at an elevated T_∞, particularly when the wall temperature T_w is higher than T_∞ (see Fig. 3).

The inaccuracy inherent in the linear viscosity law can be corrected to a certain extent when the linear approximation (4.11a) with $C \neq 1$ is used. The factor C may depend on T_w and T_∞ and also on M. One may, for example, choose C in such a way that the line (4.11a) intersects the curve (4.12a) at T_w instead of at T_∞. Then one has

$$C = \sqrt{\frac{T_w}{T_\infty}} \frac{T_\infty + T_s}{T_w + T_s} \tag{4.13}$$

This simple artifice that enters the actual computation only at the stage of equation (4.8) has been recommended by Chapman and Rubesin in

[10] In the sense of the kinetic theory of gases.

their analysis of the boundary value problem $T_w = T_w(x)$, $p_x = 0$ (19), after Chapman had applied it in (21). Then the product $\bar{\mu}\bar{\rho} = C$ would also be a function of x, but that, obviously, would have no bearing on the problem posed by equations (4.9a) and (4.9b).

When we put $\bar{\mu}\bar{\rho} = C$ in (4.9a) the variables μ and ρ are no longer "visible":

$$\frac{\partial u}{\partial \xi} = \frac{\partial}{\partial \psi}\left(u \frac{\partial u}{\partial \psi}\right) \tag{4.14}$$

But this equation could have been obtained by applying transformations (4.2) and (4.8) to equation (2.17) with *constant* μ and ρ. The solution of

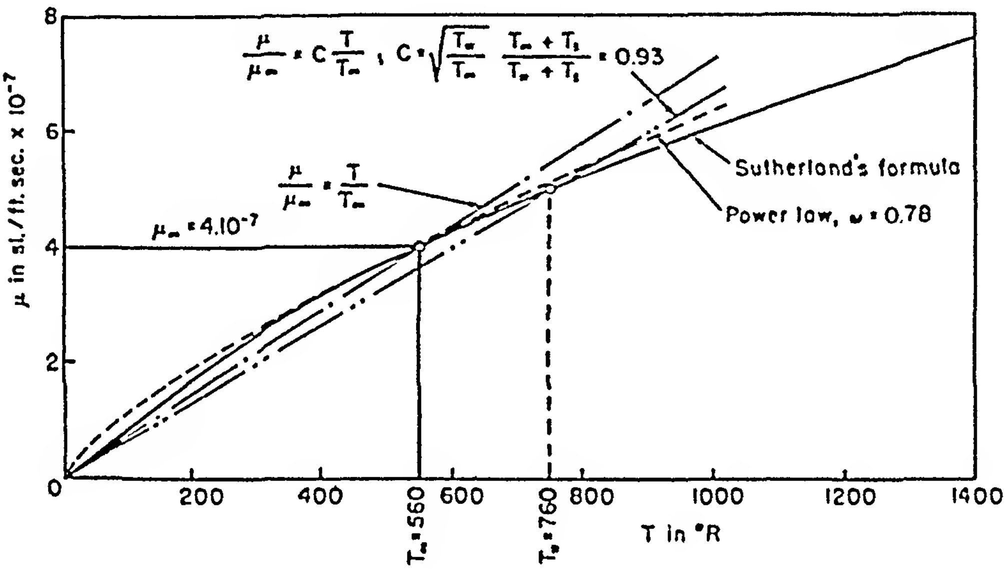


FIG. 3. Sutherland's formula, power law, and linear approximations.

(4.9a) is therefore supplied by the Blasius function $\zeta(\eta)$ (see footnote 4). Equation (4.14) becomes in fact identical with the Blasius equation on setting

$$u = \tfrac{1}{2}\zeta'(\eta) \quad \text{and} \quad \psi/\sqrt{\xi} = \zeta(\eta), \tag{4.14a}$$

and the equivalence of the boundary conditions for u is obvious. The full set of boundary conditions for (4.14) follows here for convenience:

$$\begin{aligned} u &= 0, \quad t = t_w(\xi) \quad \text{at} \quad \psi = 0 \quad (y = 0), \quad \xi \neq 0 \\ u &= 1, \quad t = 1 \quad \text{at} \quad \psi = \infty \quad (y = \infty) \end{aligned} \tag{4.15}$$

In equation (4.9b) u is now to be considered a known function, but of η. This necessitates a further change of independent variables from (ξ,ψ) to (ξ,η). The derivatives of some function $f(\xi,\psi)$ transform accord-

ing to

$$f_\xi = \frac{\partial f}{\partial \xi} - \frac{\partial f}{\partial \eta}\frac{1}{2\xi}\frac{\zeta}{\zeta'}, \qquad f_\psi = \frac{\partial f}{\partial \eta}\frac{1}{\zeta'\sqrt{\xi}} \tag{4.16}$$

where $\zeta' = d\zeta/d\eta$. Using these formulas one obtains the transformed equation (4.9b),

$$\frac{\partial^2 t}{\partial \eta^2} + \sigma\zeta\frac{\partial t}{\partial \eta} - 2\sigma\zeta'\xi\frac{\partial t}{\partial \xi} = -\frac{\sigma}{4}(\gamma - 1)M^2\zeta''^2 \tag{4.17}$$

a linear inhomogeneous differential equation for t, which is the basis of the further investigations in (19). Connection with the physical variable y can be made as follows. By (4.1) and (4.8),

$$y = \int_0^y \frac{\rho_\infty}{\rho}\frac{d\Psi}{dU} = \sqrt{\nu_\infty LC/U_\infty}\int_0^y \frac{\rho_\infty}{\rho}\frac{d\psi}{u} \quad \text{at constant } x \text{ or } \xi \tag{4.18}$$

Substituting $\rho_\infty/\rho = t$ and introducing η according to (4.14a), one obtains

$$y = \sqrt{\nu_\infty LC/U_\infty}\int_0^\eta t\cdot 2\sqrt{\xi}\,d\eta = 2\sqrt{\nu_\infty Cx/U_\infty}\int_0^\eta t\,d\eta \tag{4.18a}$$

Thus the range $0 < \eta < \infty$ is mapped on $0 < y < \infty$, which can also be concluded from (4.14a) since $\zeta(\eta) \to \infty$, with $\eta \to \infty$.

B. The Independent Variable U (Crocco, Hantzsche and Wendt). Crocco's transformation introduces the velocity U as second independent variable beside x. Instead of (4.2), we have $S = x$, $U = U(x,y)$ and because of

$$U_x = -\frac{\partial y/\partial x}{\partial y/\partial U} \quad \text{and} \quad U_y = \frac{1}{\partial y/\partial U} \tag{4.19}$$

the derivatives of some function $f(x,y)$ transform into

$$f_x = \frac{\partial f}{\partial x} - \left(\frac{\partial y/\partial x}{\partial y/\partial U}\right)\frac{\partial f}{\partial U}, \qquad f_y = \left(\frac{1}{\partial y/\partial U}\right)\frac{\partial f}{\partial U} \tag{4.20}$$

Again under the assumption $p_x = 0$, equation (2.17) transforms into

$$-\rho U\frac{\partial y}{\partial x} + \rho V = \frac{\partial}{\partial U}\frac{\mu}{\partial y/\partial U} \tag{4.21}$$

and the continuity equation (2.19) takes the form

$$U\frac{\partial y}{\partial U}\frac{\partial \rho}{\partial x} - \frac{\partial y}{\partial x}\frac{\partial(\rho U)}{\partial U} + \frac{\partial(\rho V)}{\partial U} = 0 \tag{4.22}$$

where y and V are now the *unknown* functions. The structure of these equations suggests the introduction of the viscous stress

$$\tau = \mu U_y = \frac{\mu}{\partial y / \partial U} \tag{4.23}$$

instead of y, as well as the elimination of the unknown V. This leads to the second order equation

$$U \frac{\partial}{\partial x}\left(\frac{\mu\rho}{\tau}\right) + \frac{\partial^2 \tau}{\partial U^2} = 0 \tag{4.24}$$

When the energy equation in the form (2.20a) is transformed and $\partial y/\partial U$ and ρV are again replaced according to (4.23) and (4.21), one obtains

$$\tau^2 \left(\frac{1}{c_p} + \frac{1}{\sigma}\frac{\partial^2 T}{\partial U^2}\right) + \frac{1}{2}\frac{\partial \tau^2}{\partial U}\frac{\partial T}{\partial U}\left(\frac{1}{\sigma} - 1\right) - \mu\rho U \frac{\partial T}{\partial x} = 0 \tag{4.25}$$

This form of the boundary layer equations exhibits the unknowns T and τ, that is, the physical quantities of primary practical interest. It will also be noted that equation (4.25) is of first order in τ^2 and contains only the derivative of τ^2 with respect to U. Thus a complete integral could be given for τ in terms of the derivatives of T and in terms of U, and an arbitrary function of x, which, if substituted into (4.24), would lead to a single integro-differential equation for T that describes the whole problem. It seems however, that the latter result (which is still true if p_x is retained throughout the transformation) is only of formal interest.

As before, the restriction $p_x = 0$ entails $T = T_\infty =$ constant for $u = U_\infty$. The corresponding boundary condition for the stress τ is obviously $\tau = 0$. At $U = 0$ we choose[11] $T_w =$ constant but $\partial\tau/\partial U$ must needs vanish there [put $U = V = p_x = 0$ in (2.17), then $0 = \tau_y$, which is $(\tau/\mu)\partial\tau/\partial U$ by (4.20) and (4.23)]. As in (4.15), the singular point $x = 0$ must be excluded.

The next step in Crocco's analysis must correspond to the substitution (3.17) of (11) or (4.14a) of (19). In the present case we use the fact that T is a function of U alone not only in the simple, but also in the restricted problem. This, quite obviously, must be so, since both T and U are functions of $y/\sqrt{x}$ (see footnote 7), and becomes apparent if one introduces

$$\tau = f_1(x) f_2(U) \tag{4.26}$$

[11] One might call this the *restricted* problem. It is identical with what was called previously the simple problem except for $\sigma \neq 1$.

in (4.24) and (4.25) and puts $\partial T/\partial x = 0$. One then obtains[12]

$$f_1(x) = \frac{1}{\sqrt{2x}} \tag{4.27}$$

hence $f_2(U) = \tau\sqrt{2x}$, and the following system of ordinary differential equations for f_2 and T results:

$$f_2 f_2'' + \mu\rho U = 0 \tag{4.28a}$$
$$(c_p T'' + \sigma) f_2 + (1 - \sigma) c_p T' f_2' = 0 \tag{4.28b}$$

where the prime denotes differentiation with respect to U.

Introducing dimensionless variables u, t, $\bar{\rho}$, $\bar{\mu}$ as before and the stress field coefficient

$$c_\tau = \frac{2\tau}{\rho_\infty U_\infty^2} = \frac{2f_2(U)}{\rho_\infty U_\infty^2\sqrt{2x}} \tag{4.29}$$

we define a dimensionless function of u

$$K(u) = c_\tau\sqrt{\mathrm{Re}}\sqrt{x/L} \quad \left(= \sqrt{\frac{2}{\rho_\infty \mu_\infty U_\infty^3}}\, f_2(U) \right) \tag{4.30}$$

to replace $f_2(U)$. Then the system (4.28) reads

$$KK'' + 2\bar{\mu}\bar{\rho}u = 0 \tag{4.31a}$$
$$[t'' + \sigma(\gamma - 1)M^2]K + (1 - \sigma)t'K' = 0 \tag{4.31b}$$

the primes now denoting differentiation with respect to u. The dimensionless boundary conditions are

$$\begin{aligned} K' &= 0, \quad t = T_w/T_\infty, \quad \text{at } u = 0 \\ K &= 0, \quad t = 1, \qquad\quad\ \ \text{at } u = 1 \end{aligned} \tag{4.32}$$

The total friction force F can be calculated by a simple integration over x,

$$F = \int_0^L \tau_0 dx = \frac{\rho_\infty U_\infty^2 L K_0}{\sqrt{\mathrm{Re}}} \tag{4.33}$$

where τ_0 and K_0 stand for $\tau_{u=0}$ and $K_{u=0}$. Equation (4.33) is equivalent to (3.20). The parameter K_0 introduced in (3.20) appears now as the initial value of the basic dynamic variable $K(u)$ in Crocco's analysis. The return to the field variables x, y is effectuated by integrating $\tau = \mu(\partial U/\partial y)$ at constant x:

$$y(x,U) = \int_0^U \frac{\mu dU}{\tau} \tag{4.34}$$

[12] Equation (4.24) yields $[1/f_1(x)]d[1/f_1(x)]/dx = -\{f_2(U)d^2[f_2(U)]/dU^2\}/\mu\rho U$, which must be a constant (say, 1) since μ and ρ depend only on U through T.

Substituting for τ according to (4.30) one has

$$y = 2\sqrt{\frac{xL}{\mathrm{Re}}}\int_0^u \frac{\bar{\mu}\,du}{K} \quad \text{or} \quad \eta = \int_0^u \frac{\bar{\mu}\,du}{K} \tag{4.35}$$

which determines the Blasius variable η in terms of the solution of the transformed equations.

Equations (4.24, 25) are also the starting point of Hantzsche and Wendt (13). In their notation, $\tau = H(x)G(U)$ corresponding to (4.26), and $H(x) = 1/\sqrt{x}$ corresponding to (4.27), so that $G = f_2/\sqrt{2}$ and $H = f_1\sqrt{2}$. The further analysis is based on a viscosity power law $\mu \propto T^\omega$ where ω is so chosen that T^ω touches the Sutherland curve (4.12) at $T = T_w$. This leads to

$$\frac{\mu}{\mu_w} = \left(\frac{T}{T_w}\right)^\omega, \qquad \omega = \frac{3}{2} - \frac{1}{1 + T_s/T_w} \tag{4.36}$$

On introducing the usual dimensionless u, but $g(u)$ and j by

$$\tau\sqrt{x} = G(U) = \sqrt{\frac{\mu_w\rho_w U_\infty^3}{2}}\,g(u) \quad \text{and} \quad c_pT = U_\infty^2 j, \tag{4.37}$$

one obtains as equivalent form of (4.28):

$$gg'' + u\left(\frac{j_w}{j}\right)^{1-\omega} = 0 \tag{4.38a}$$

$$(j'' + \sigma)g + (1 - \sigma)g'j' = 0 \tag{4.38b}$$

with boundary conditions corresponding to (4.32).

The condition of vanishing heat transfer

$$\frac{\partial t}{\partial y} = 0 \quad \text{or} \quad \frac{\partial j}{\partial y} = 0 \text{ at } y = 0 \tag{4.39}$$

reads in the variable u

$$t' = 0 \quad \text{or} \quad j' = 0 \quad \text{at } u = 0 \tag{4.40}$$

[cf. equations (4.20) and (4.23)]. The relation

$$g(u)\sqrt{2} = \sqrt{\frac{\mu_\infty\rho_\infty}{\mu_w\rho_w}}\,K(u) = \left(\frac{T_\infty}{T_w}\right)^{(\omega-1)/2} K(u) \tag{4.41}$$

yields immediately the equations corresponding to (4.33) and (4.35). The latter reads now

$$y = \sqrt{\frac{2x\mu_w}{U_\infty\rho_w}}\int_0^u \left(\frac{T}{T_w}\right)^\omega \frac{du}{g} \tag{4.42}$$

C. The Stretched y Coordinate (Howarth). Howarth's transformation has been devised to attack the boundary layer problem in the case of nonvanishing pressure gradient. But as Cope and Hartree point out (18), analysis in general terms must be abandoned at a very early stage, unless the introduction of the term p_x is accompanied by a considerable simplification of the other terms. Thus Howarth adopts the assumption $\sigma = 1$ and puts $\mu \propto T$; the product $\mu\rho$ is no longer constant, but depends on x.

Since ρ_∞ is not a constant either, we refer the stream function Ψ and the functions ρ and μ to some standard state T_s, ρ_s, μ_s, to be specified later. Thus we have

$$\sigma = 1, \qquad \frac{\mu}{\mu_s} = \frac{T}{T_s}, \qquad \frac{\mu\rho}{\mu_s\rho_s} = \frac{p}{p_s} \tag{4.43}$$

The subscript ∞ in (4.1) and (4.7) is to be replaced by s. The same applies to the equation of motion (4.6), and its second member must be augmented by the term $-p_x/\rho U$. Since we have inviscid flow at the boundary layer edge,

$$-\frac{1}{U\rho}p_x = \frac{1}{U\rho}\left(\rho_\infty U_\infty \frac{dU_\infty}{dx}\right) = \frac{T}{T_\infty}\frac{U_\infty}{U}\frac{dU_\infty}{dx} \tag{4.44}$$

where (U, ρ, T) and $(U_\infty, \rho_\infty, T_\infty)$ refer to the same x-coordinate, of course. The augmented equation (4.6) reads

$$\frac{\partial U}{\partial x} = \frac{T}{T_\infty}\frac{U_\infty}{U}\frac{dU_\infty}{dx} + \nu_s \frac{p}{p_s}\frac{\partial}{\partial\Psi}\left(U\frac{\partial U}{\partial\Psi}\right) \tag{4.45}$$

since p does not depend on Ψ.

With the idea of simplifying the "viscous" term and removing the factor p/p_s, we introduce a new independent variable Y by setting

$$\frac{\partial}{\partial\Psi} = \left(\frac{p}{p_s}\right)^{-1/2}\frac{1}{U}\frac{\partial}{\partial Y} \tag{4.46}$$

that is, we transform variables x, Ψ into S, Y by

$$S = x, \qquad Y = \left(\frac{p}{p_s}\right)^{-1/2}\int_0^\Psi \frac{d\Psi}{U} \quad \text{(a length)} \tag{4.46a}$$

Using, as on p. 38, subscripts and ∂-notation to distinguish between "old" and "new" derivatives,[13] we obtain for some function f

$$f_x = \frac{\partial f}{\partial x} - \frac{\partial f}{\partial Y}\frac{\partial\Psi}{\partial x}\frac{1}{U}\left(\frac{p}{p_s}\right)^{-1/2} \tag{4.47}$$

[13] Derivatives in (4.45) should now be redenoted by subscripts.

while the right member of (4.46) gives f_ψ. Then the transformed equation (4.45) reads after multiplication with U

$$U\frac{\partial U}{\partial x} - \frac{\partial U}{\partial Y}\left(\frac{p}{p_s}\right)^{-\frac{1}{2}}\frac{\partial \Psi}{\partial x} = U_\infty \frac{dU_\infty}{dx}\frac{T}{T_\infty} + \nu_s \frac{\partial^2 U}{\partial Y^2} \tag{4.48}$$

Consider now the term with $\partial\Psi/\partial x$. According to (4.46), $U = (p/p_s)^{-\frac{1}{2}}\partial\Psi/\partial Y$, where Ψ is now considered as a function of x, Y. But p being independent of Y, we can now introduce a quantity

$$\chi(x,Y) = \left(\frac{p}{p_s}\right)^{-\frac{1}{2}}\Psi(x,Y) \tag{4.49}$$

that plays the role of an "incompressible" stream function in the "distorted" plane x, Y (but only for the component U) since obviously

$$U = \frac{\partial\chi(x,Y)}{\partial Y} \tag{4.50}$$

For the term $\partial\Psi/\partial x$ we obtain

$$\frac{\partial \Psi}{\partial x} = \left(\frac{p}{p_s}\right)^{\frac{1}{2}}\frac{\partial\chi}{\partial x} - \frac{p_x}{2p_s}\left(\frac{p}{p_s}\right)^{-\frac{1}{2}}\chi = \left(\frac{p}{p_s}\right)^{\frac{1}{2}}\left(\frac{\partial\chi}{\partial x} - \frac{p_x}{2p}\chi\right) \tag{4.51}$$

In (4.48) we now replace U and $\partial\Psi/\partial x$ according to (4.50) and (4.51) and observe that by (4.44)

$$\frac{p_x}{p} = -U_\infty\frac{dU_\infty}{dx}\frac{\rho_\infty}{p_\infty} = U_\infty\frac{dU_\infty}{dx}\frac{\gamma}{a_\infty^2} \tag{4.52}$$

where a_∞, a function of x, is the local velocity of sound at the boundary layer edge. Then (4.48) takes the form

$$\frac{\partial\chi}{\partial Y}\frac{\partial^2\chi}{\partial x\partial Y} - \frac{\partial^2\chi}{\partial Y^2}\frac{\partial\chi}{\partial x} = U_\infty\frac{dU_\infty}{dx}G + \nu_s\frac{\partial^3\chi}{\partial Y^3} \tag{4.53}$$

where

$$G = \frac{T}{T_\infty} - \frac{\gamma}{2a_\infty^2}\chi\frac{\partial^2\chi}{\partial Y^2}$$

It will be noticed that transformation (4.46) is, in a sense, inverse to the Mises transformation (4.4). Hence we could have started with equation (4.6a), applying first the transformation equivalent to (4.46a)

$$Y = \int_0^y \left(\frac{\nu_s}{\nu}\right)^{\frac{1}{2}} dy = \left(\frac{p}{p_s}\right)^{\frac{1}{2}}\int_0^y \frac{T_s}{T}\,dy \tag{4.54}$$

and then (4.49). This, actually, has been Howarth's representation (17). Doubtlessly the relation between the x, y- and x, Y-planes is better char-

acterized when (4.54) is used in the first place. Starting from (4.6) requires, on the other hand, somewhat less "algebra" and gives the single steps perhaps a more compelling appearance.

Since $\sigma = 1$, and only the boundary condition $T_y = 0$ will be considered, the general integral of the energy equation is

$$T^* = \text{constant} \qquad [(3.9')]$$

This implies

$$2c_pT_1 + U_1^2 = 2c_pT_2 + U_2^2 \qquad (4.55')$$

valid for *any* pair of points (1), (2), but when we apply (4.55′) to pairs (x,y), (x,∞), then it becomes a relation between two functions of (x,y) and two functions of x and may be written

$$\frac{T}{T_\infty} = 1 + \frac{\gamma - 1}{2a_\infty^2}(U_\infty^2 - U^2) = 1 + \frac{\gamma - 1}{2} M_\infty^2(1 - u^2) \qquad (4.55)$$

Substitution in equation (4.53) makes the compressibility factor G appear in the form

$$G = 1 + \frac{\gamma - 1}{2a_\infty^2}\left[U_\infty^2 - \left(\frac{\partial\chi}{\partial Y}\right)^2\right] - \frac{\gamma}{2a_\infty^2}\chi\frac{\partial^2\chi}{\partial Y^2} \qquad (4.56)$$

In the incompressible case, $a_\infty = \infty$ and $G = 1$.

Thus Howarth's transformation presents the equation of motion in a shape that is formally identical with the Prandtl-Blasius equation of the incompressible case (4.6a), the entire influence of compressibility being concentrated in the factor G of the pressure term. For *uniform* flow (simple problem) equation (4.53) is altogether identical with the corresponding incompressible problem, and the compressible solution can be obtained by subjecting the plane x, Y of the associated incompressible flow to the (inverse) transformation (4.54).

2. Transformations Adapting Compressible Boundary Layer Equations to Application of Approximate Methods Developed for Incompressible Boundary Layer

A. The Stretched y-Coordinate (Kalikhman)—Generalization of Pohlhausen's Method by Kalikhman. In Kalikhman's analysis (20) the unknown function T is replaced by the stagnation temperature T^* of equation (3.3). The value of T^* at the boundary layer edge is a given constant T_∞^* which may be used as a reference temperature also in the general case $p_x \neq 0$. We now admit $T_w = T_w(x)$, denote the dimensionless temperature by $t = T/T_\infty^*$ as before, introduce the dimensionless stagnation temperature $t^* = T^*/T_\infty^*$ and the excess of the stagnation

temperature over the wall temperature at the same x-section

$$E^* = T^* - T_w \quad \text{with} \quad E_\infty^* = T_\infty^* - T_w \quad \text{and} \quad E_w^* = 0 \tag{4.57}$$

Then

$$t^* = 1 - \frac{E_\infty^* - E^*}{T_\infty^*} = 1 + \frac{T_w - T_\infty^*}{T_\infty^*}\frac{E_\infty^* - E^*}{E_\infty^*} \tag{4.58}$$

and the dimensionless temperature can be expressed in terms of U, E^*, and the given quantities t_w and E_∞^*:

$$t = 1 - \frac{U^2}{2c_pT_\infty^*} + (t_w - 1)\left(1 - \frac{E^*}{E_\infty^*}\right) \tag{4.59}$$

This representation of t is valid everywhere in the boundary layer; for the boundary layer edge we have in particular

$$t_\infty = 1 - \frac{U_\infty^2}{2c_pT_\infty^*} \tag{4.59$'$}$$

Under certain conditions equation (4.59) assumes the form of Crocco's integral. Indeed, on introducing the boundary condition $t_w = 1$ in (4.59) one obtains a relation that is equivalent to (3.9′) namely $T^* = T_\infty^*$. If this is substituted in the energy equation (3.6), there results $(\sigma^{-1} - 1)\partial(\mu T_y)/\partial y = 0$. It follows that one must set $\sigma = 1$. We conclude therefore that the boundary condition $t_w = 1$ or $T_w = T_\infty^*$, often called complete temperature recovery, can only be fulfilled in a fluid with $\sigma = 1$; it obviously implies $(T_y)_w = 0$. The boundary condition $t_w = \text{constant} \neq 1$, however, leads to Crocco's integral (3.9) only if U_∞ is constant and $E^*/E_\infty^* = U/U_\infty$, a relation which in itself is equivalent to (3.9).

It is convenient to write

$$\bar{U}^2 = \frac{U^2}{2c_pT_\infty^*}$$

Then the pressure is

$$p = p_S(1 - \bar{U}_\infty^2)^{\gamma/(\gamma-1)} \tag{4.60}$$

density and viscosity are known in terms of U and E^*,

$$\rho = \rho_S(1 - \bar{U}_\infty^2)^{\gamma/(\gamma-1)}/t \tag{4.61a}$$
$$\rho_\infty = \rho_S/(1 - \bar{U}_\infty^2)^{1/(\gamma-1)} \tag{4.61b}$$
$$\mu = \mu_S t^\omega \text{ (say)} \tag{4.61c}$$

where $\bar{U}_\infty(x)$ is a given function, p_S and ρ_S are the stagnation values of p and ρ along the boundary layer edge, $\mu_S = \mu(T_\infty^*)$, and t is given by (4.59).

only. Then (3.6) transforms into

$$U \frac{\partial E^*}{\partial x} + \tilde{V} \frac{\partial E^*}{\partial \eta} - U \frac{dE_\infty^*}{dx} = \nu_S \frac{\partial}{\partial \eta}\left(\frac{\rho}{\rho_S} t^\omega \frac{\partial E^*}{\partial \eta}\right) \tag{4.67}$$

The viscous terms in (4.66) and (4.67) exhibit the same second order "operator," which we can write in the form

$$D^{(2)} \equiv \nu_S(1 - \bar{U}_\infty^2)^{\gamma/(\gamma-1)} \frac{\partial}{\partial \eta} t^{\omega-1} \frac{\partial}{\partial \eta}$$

it is applied to U and E^* respectively.

We now turn to Kalikhman's generalization of Pohlhausen's method[15] for the case of the compressible boundary layer. As is well known, Pohlhausen replaces the actual velocity profile $u(x,y)$ by a quartic in y/δ, where δ is the thickness of the boundary layer. "In the approximate solution δ is to be regarded as a convenient parameter rather than as a dependent variable whose value is sought" [(1), p. 158, where Pohlhausen's method is reviewed]. In the present case *two* such thickness parameters are needed, namely,

$$\delta = \text{dynamic b.l. thickness}, \quad \Delta = \text{thermal b.l. thickness}$$

both are functions of x.

It is also convenient to introduce $u = U/U_\infty$ and a dimensionless excess stagnation temperature

$$c^* = \frac{E^*}{E_\infty^*} = \frac{T^* - T_w}{T_\infty^* - T_w} \tag{4.68}$$

The approximating profiles of the velocity and excess stagnation temperature distributions are now assumed as

$$(1)\quad u = \sum_1^4 A_n(\eta/\delta)^n \qquad (2)\quad c^* = \sum_1^4 B_n(\eta/\Delta)^n \tag{4.69}$$

where η is the variable defined in (4.62). The eight coefficients are determined in the same way as in the incompressible case. Six linear equations are obtained if one requires u and c^* to approach the value 1 at $\eta/\delta = 1$ $(\eta/\Delta = 1)$, and this so smoothly that first *and* second derivatives with respect to η/δ (η/Δ) vanish. Two further conditions reflect

[15] E. Pohlhausen, *Z. angew. Mathem. Mech.*, **1**, 257 (1921). See Also *NACA*, *Tech. Mem.* No. **1217**, p. 83ff., which is a translation of a course of lectures given by H. Schlichting in 1941 at Braunschweig. The method has been systematized by Howarth in 1935 (*A.R.C. Repts. and Memo.* No. **1632**).

the behavior of U and $\tilde{V}$ at $\eta = 0$: the first members of equations (4.66) and (4.67) vanish at $\eta = 0$ (corresponding to $y = 0$), which implies

$$\left.\begin{aligned} D^{(2)}U &= -\frac{\rho_\infty}{\rho} U_\infty U_\infty' \\ D^{(2)}E^* &= 0 \end{aligned}\right\} \quad \text{at } \eta = 0. \qquad \begin{aligned}(4.70a)\\(4.70b)\end{aligned}$$

Because neither U_∞ nor $E_\infty{}^*$ depend on η, one may rewrite these equations as

$$\left.\begin{aligned} D^{(2)}u &= -\frac{\rho_\infty}{\rho} U_\infty' \\ D^{(2)}e^* &= 0 \end{aligned}\right\} \quad \text{at } \eta = 0 \qquad \begin{aligned}(4.71a)\\(4.71b)\end{aligned}$$

substitute the approximative profiles (4.69) for u and e^*, carry out the differentiations in (4.71), and finally put $\eta = 0$. Then a linear relation between A_2 and A_1B_1 and another one between B_2 and $B_1{}^2$ is obtained; the first one, developed from (4.71a), involves an expression which corresponds to the conventional form parameter $\Lambda = -d^2u/[d(y/\delta)]^2$ at $y = 0$, namely,

$$\lambda = \delta^2 U_\infty' t_w{}^{2-\omega}/\nu s(1 - \bar{U}_\infty{}^2)^{1+\gamma/(\gamma-1)} \tag{4.72a}$$

and in addition a known function of x, namely,

$$\alpha = (1 - \omega)(1 - 1/t_w) \tag{4.72b}$$

The second one, developed from (4.71b), involves only α.

The six linear relations serve to express the $A_{2,3,4}$ by A_1 and the $B_{2,3,4}$ by B_1 exactly as in the incompressible case and lead, of course, to the same formulas. Then one calculates from conditions (4.71)

$$A_1 = \frac{12 + \lambda}{6 - (3 - \sqrt{9 - 12\alpha})\delta/\Delta}, \qquad B_1 = \frac{3 - \sqrt{9 - 12\alpha}}{\alpha} \tag{4.73}$$

We now have to set up the relations that take the place of the differential equation for the momentum thickness

$$\frac{d\theta^{\text{inc}}}{dx} + \frac{U_\infty'}{U_\infty}(2\theta^{\text{inc}} + \delta_1{}^{\text{inc}}) = \frac{\tau_w}{\rho U_\infty{}^2} \tag{4.74}$$

with

$$\theta^{\text{inc}} = \int^\delta u(1 - u)dy, \quad \delta_1{}^{\text{inc}} = \int_0^\delta (1 - u)dy$$

in the incompressible case. Equation (4.64) and the left member of equation (4.66) have the "incompressible" form, hence we obtain by

integrating (4.66) over η from 0 to ∞ [(1), p. 132]

$$\frac{\partial}{\partial x}\int_0^\infty U^2 d\eta - U_\infty \frac{\partial}{\partial x}\int_0^\infty U d\eta = U_\infty U_\infty' \int_0^\infty \frac{\rho_\infty}{\rho} d\eta - \frac{\tau_w}{\rho s} \qquad (4.75)$$

where τ_w is the stress at the wall. Upon substituting for ρ_∞/ρ according to (4.61a), (4.61b) and (4.59) and some rearranging, it is possible to express (4.75) by the following three integral parameters:

$$\theta = \int_0^\delta u(1 - u)d\eta \quad \text{momentum thickness} \qquad (4.76a)$$

$$\delta_1 = \int_0^\delta (1 - u)d\eta \quad \text{displacement thickness} \qquad (4.76b)$$

$$\Delta^* = \int_0^\Delta (1 - c^*)d\eta \quad \text{thermal mixing thickness} \qquad (4.76c)$$

(The limit ∞ has been replaced by δ and Δ respectively, as usual.)

When we introduce the abbreviations

$$\varphi_1 = \frac{\bar{U}_\infty^2}{(1 - \bar{U}_\infty^2)} = \frac{1}{2}(\gamma - 1)M_\infty^2, \qquad \varphi_2 = \frac{t_w - 1}{1 - \bar{U}_\infty^2} = \frac{t_w - 1}{t_\infty}$$

where M_∞ is the variable local Mach number along the boundary layer edge, equation (4.75) becomes finally

$$\frac{d\theta}{dx} + \frac{U_\infty'}{U_\infty}[(\varphi_1 + 2)\theta + (\varphi_1 + 1)\delta_1 + \varphi_2\Delta^*] = \frac{\tau_w}{\rho s U_\infty^2} \qquad (4.77)$$

The energy equation (4.67) is now integrated in the same manner. We first rewrite, using (4.64),

$$U(\partial E^*/\partial x) + \tilde{V}(\partial E^*/\partial \eta) = \partial(UE^*)/\partial x + \partial(\tilde{V}E^*)/\partial\eta$$

and

$$-U(dE_\infty^*/dx) = -\partial(UE_\infty^*)/\partial x - E_\infty^*(\partial\tilde{V}/\partial\eta)$$

Then (4.67) integrates to

$$\frac{d}{dx}\int_0^\infty U(E^* - E_\infty^*)d\eta = \left[\frac{\mu}{\rho s k} kT_y^*\right]_0^\infty = \frac{q_w}{\rho s c_p} \qquad (4.78')$$

since $\sigma = 1$.[16] Upon introducing a fourth integral parameter

$$\Theta = \int_0^\Delta u(1 - c^*)d\eta \quad \text{energy thickness} \qquad (4.76d)$$

[16] Equation (4.77) is actually a general form of the momentum equation, since $\sigma = 1$ has not been used in its derivation; also, equation (4.78') remains correct if $\sigma \neq 1$. Yet the method seems to work only under the condition $\sigma = 1$, which is responsible for the relatively simple form of the crucial relations (4.73).

equation (4.78′) takes the form

$$\frac{d\Theta}{dx} + \frac{U_\infty'}{U_\infty}\Theta + \frac{E_\infty^{*\prime}}{E_\infty^*} = \frac{q_w}{\rho s U_\infty c_p E_\infty^*} \tag{4.78}$$

Equations (4.77) and (4.78) represent two momentum equations of the compressible boundary layer, one corresponding to the integrated equation of motion, the other to the integrated energy equation.[17]

The further steps in Kalikhman's paper follow closely the conventional Pohlhausen method. The stress τ_w and the unit heat transfer q_w can be expressed by the first derivatives of the polynomials (4.69) at $\eta = 0$, that is, by $\tau_w \propto A_1$ and $q_w \propto B_1$; the factors of proportionality depend on the given boundary functions t_w and U_∞, on ω, and on δ and Δ respectively. The four integral parameters (4.76) must be evaluated in terms of A_1, B_1, δ and Δ; the corresponding formulas are given in (20). (Note that for Θ *two* expressions are obtained, depending on whether $\Delta < \delta$ or $\Delta > \delta$ is assumed.)

Since λ is known in terms of δ by (4.72a), equations (4.77) and (4.78) represent two simultaneous differential equations for two of the boundary layer parameters, say δ and Δ. The question of the initial conditions for these equations is discussed in (20) as usually, that is, in connection with the existence of a forward stagnation point ($x = 0$) at a cylindrical obstacle, and the variation of λ_{stag} ($= 7.052$ in the incompressible case) with $(t_w)_{x=0}$ is tabulated.[18] To obtain a practical solution of the system (4.77) and (4.78) a method of successive approximations is outlined in the form of a generalization of the method developed by Miss Lyon.[19]

Unfortunately, Kalikhman's paper contains no numerical details with the exception of a diagram showing results obtained for a lens-shaped profile (Nu vs. x/c at $M = 2$ and 6 for $t_w = 0\,25$). Thus it is impossible to balance the amount of numerical work against the accuracy obtained. It seems also doubtful whether the quartic approximation of the e^*-profile is better than an approximate solution of the energy equation that might be developed on the basis of the Crocco integral (cf. the remark in (18), end of sec. 3b, for the case $(T_y)_w = 0$). Note further a similar attempt by Young (25, sec. 3) who avoids the use of a second relation such as (4.78) by reasonable assumptions about the quantities that correspond in his analysis to δ_1/θ and δ/θ.

[17] Also the terms *momentum integral equation* and *energy integral equation* are in frequent use. A different form of (4.78) is found in (1), p. 613ff.; cf. also F. Frankl, *CAHI Report* No. 176 (1934) (Russian) equation (2.10) and the preceding one. Another form of (4.77) is found in the following section IV, 2B.

[18] According to the first equation (4.73), $\lambda = -12$ determines the separation point ($\tau_w = 0$) as in the incompressible case.

[19] H. Lyon, The drag of streamlined bodies. Aircraft Engineering 6, 233 (1934).

B. Generalization of Pohlhausen's Method by Howarth. Howarth, in (17), also developed a generalization of Pohlhausen's method. Recalling the subject of that paper which is the case $\sigma = 1$, $\mu \propto T$, $p_x \neq 0$, $(T_y)_w = 0$, we can set up the momentum equation by specializing (4.77). Denoting the momentum and displacement thickness with respect to the variable Y [equation (4.54)] by θ' and δ_1' respectively[20] and observing that φ_2 in (4.77) vanishes for the present case, we rewrite (4.77) in terms of Y. When the (freely choosable) standard subscript s in Eq. (4.54) is temporarily identified with the stagnation subscript S, equations (4.54) and (4.62) imply

$$d\eta = \left(\frac{p}{p_S}\right)^{1/2} dY, \quad \text{whence} \quad \theta' = \left(\frac{p}{p_S}\right)^{1/2} \theta, \quad \delta_1' = \left(\frac{p}{p_S}\right)^{1/2} \delta_1, \tag{4.79}$$

and Eq. (4.77) takes the form

$$U_\infty{}^2 \frac{d\theta'}{dx} + U_\infty \frac{dU_\infty}{dx} [\theta'(2 - \tfrac{1}{2}M_\infty{}^2) + \delta_1' (1 + \tfrac{1}{2}(\gamma - 1)M_\infty{}^2)] = \nu_S \left(\frac{\partial U}{\partial Y}\right)_w. \tag{4.80}$$

The old notation s may now be restored in (4.80) and (4.54), since S occurs in both equations solely as subscript of ν.

It will be noted that integral parameters such as θ and θ' do not transform into one another [cf. (4.79)] and thus have no independent physical meaning. They represent the "momentum thickness" with respect to a certain coordinate (η or Y). If one introduces the "natural" momentum and displacement thicknesses

$$\theta^c = \int_0^{\delta^c} \frac{U\rho}{U_\infty \rho_\infty}\left(1 - \frac{U}{U_\infty}\right) dy \quad \text{and} \quad \delta_1{}^c = \int_0^{\delta^c} \left(1 - \frac{U\rho}{U_\infty \rho_\infty}\right) dy \tag{4.81'}$$

one obtains either by transformation of equation (4.77) or by integration of the original equation of motion (2.17) and use of (2.19) the following form of the momentum equation

$$\frac{d\theta^c}{dx} + \frac{1}{U_\infty}\frac{dU_\infty}{dx}[(2 - M_\infty{}^2)\theta^c + \delta_1{}^c] = \frac{\tau_w}{\rho_\infty U_\infty{}^2} \tag{4.81}$$

Returning to equation (4.80) which is a particular case of (4.81), we note that there is, of course, no need for a counterpart of equation (4.78) since the correct integral of the energy equation now is known and has already been used[21] in setting up (4.80). The conventional form

[20] $\theta' = \int_0^{\delta'} u(1 - u)dY$, $\delta_1' = \int_0^{\delta'} (1 - u)dY$.

[21] In the form of $\varphi_2 = 0$ or $t_w = 1$, which is equivalent to $T^* =$ constant (cf. p. 49).

of the Pohlhausen method may therefore be used, that is, one expresses the quartic by which u is approximated in terms of the two quartics[22] $F_1(Y/\delta')$ and $F_2(Y/\delta')$ and the form parameter λ:

$$u = \frac{U}{U_\infty} = F_1 + \lambda F_2 \tag{4.81a}$$

As before, the equation of motion (4.53), applied at $Y = 0$, yields the condition for λ. One has

$$\left[\frac{\partial^3 \chi}{\partial Y^3}\right]_w = -\frac{1}{\nu_S} G_w U_\infty \frac{dU_\infty}{dx} \tag{4.82}$$

and by (4.81a) and (4.50)

$$\left[\frac{d^2 u}{d(Y/\delta')^2}\right]_w = -\lambda = \frac{\delta'^2}{U_\infty}\left[\frac{\partial^3 \chi}{\partial Y^3}\right]_w$$

where δ' is the boundary layer thickness in the coordinate Y. Consequently

$$\lambda = \frac{\delta'^2}{\nu_S}\frac{dU_\infty}{dx} G_w, \qquad G_w = 1 + \frac{\gamma - 1}{2} M_\infty{}^2 \tag{4.83}$$

Equation (4.83) differs from the corresponding incompressible equation only by the factor G_w.

The integrals θ' and δ_1' can be evaluated in terms of δ' and, by (4.83), in terms of λ. Substitution in (4.80) finally yields the first order differential equation for λ that determines the problem. It reads, when derivation with respect to x is denoted by a prime,

$$\lambda' = \frac{U_\infty''}{U_\infty'}(\lambda^2 h + \lambda) + \frac{U_\infty'}{U_\infty}\{g + M_\infty[\gamma(\lambda^2 h + \lambda) + \tfrac{1}{2}(\gamma - 1)j]\} \tag{4.84}$$

where $g(\lambda)$, $h(\lambda)$, $j(\lambda)$ are certain rational functions of λ tabulated in (1,17).

Howarth applies equation (4.84) to estimate the influence of compressibility on the separation point in retarded flow. If a linear velocity lapse $U_\infty = U_0 - xU_1$ is assumed λ can be found by a quadrature, provided $M_\infty(x)$ is replaced by an average value $\bar{M}$. In this case the separation point abscissa x_{sep} is found to move *upstream* with increasing Mach number. This is seen from the velocity drop between 0 and x_{sep} which is 12% in the incompressible case if computed by power series methods; it decreases in the present case from 15.6% to 4.4% when the Mach number at the leading edge $x = 0$ increases from 0 to 10. We shall return to other results of (17) in the section on results.

[22] See (1), p. 158 for F_1 and F_2. Note that $F''(\theta) = 0$ and $G''(\theta) = -1$.

A thorough analysis of the general use of Pohlhausen's method in compressible boundary layer problems is found in (18), section 4a. The case under consideration is $\sigma \neq 1$ (but of the order of 1), $p_x = 0$, and $(T_y)_w = 0$. It is assumed that the energy equation is integrated with enough accuracy by $T^* = constant$ so that the factor ρ/ρ_∞ in θ^c and δ_1^c [see (4.81′)] can be expressed in terms of M_∞ and u by means of (4.55). The velocity distribution is again assumed in the form $u = F_1 + \lambda F_2$ as in (4.81a), but the functions $F_1(y/\delta^c)$ and $F_2(y/\delta^c)$ are left undetermined. Under these fairly general conditions Cope and Hartree develop from (4.81) the differential equation for λ which is the analogue of equation (4.84). This equation, however, is not yet explicit in λ; the integrals θ^c and δ_1^c can be written in terms of the following two integrals

$$\int_0^1 = \frac{d(y/\delta^c)}{1 \pm \sqrt{\alpha}\, u}, \qquad \alpha = \frac{\varphi_1}{1 + \varphi_1} \tag{4.85}$$

but these should be explicitly evaluable in λ, to make the method practically useful, that is, applicable without too early recourse to numerical methods. This condition, however, restricts the choice of functions F_1 and F_2 essentially to quadratic polynomials; even so the equation for λ becomes complicated enough, and one must accept the low level of accuracy inherent in the quadratic approximation of u.

3. Transformations That Prepare the Equations for a Numerical Computer

A. Series Expansions (Howarth). Howarth in (17) and particularly Cope and Hartree in (18) have extended the method of *series expansion*, so well established in incompressible boundary layer theory, to the present problem. A first insight into the characteristic difficulties can be obtained by studying Howarth's treatment of the *linearly accelerated flow*. The leading assumptions have been given at the beginning of section IV, 1C. The starting point is equation (4.53) with

$$U_\infty = \beta x, \qquad U_\infty \frac{dU_\infty}{dx} = \beta^2 x,$$
$$G = 1 + \frac{\gamma - 1}{2a_\infty^2}\left[\beta^2 x^2 - \left(\frac{\partial \chi}{\partial Y}\right)^2\right] - \frac{\gamma}{2a_\infty^2}\,\chi\,\frac{\partial^2 \chi}{\partial Y^2} \tag{4.86}$$

The solution of this problem in the incompressible case is completely known (flow in the vicinity of a stagnation point [see (1), p. 139]); one chooses, of course, a method that takes advantage of that knowledge.

We first express a_∞ in terms of the given U_∞,

$$a_\infty^2 + \frac{\gamma - 1}{2}\beta^2 x^2 = a_s^2 \tag{4.87}$$

and identify the standard state s in (4.53) and (4.54) with the state of stagnation along the boundary layer edge. Now we expand

$$\frac{1}{a_\infty^2} = \frac{1}{a_S^2}\left(1 + \frac{\gamma - 1}{2}\frac{\beta^2 x^2}{a_S^2} + \cdots\right) \tag{4.88}$$

and introduce this expansion in expression (4.86). The term $G\beta^2 x$ then takes the following form:

$$\begin{aligned} G\beta^2 x = {} & \beta^2 x\left[1 - \frac{\gamma - 1}{2a_S^2}\left(\frac{\partial\chi}{\partial Y}\right)^2 - \frac{\gamma}{2a_S^2}\chi\frac{\partial^2\chi}{\partial Y^2}\right] \\ & + \beta^4 x^3\left\{\frac{\gamma - 1}{2a_S^2} - \frac{(\gamma - 1)^2}{4a_S^4}\left(\frac{\partial\chi}{\partial Y}\right)^2 - \frac{\gamma(\gamma - 1)}{4a_S^4}\chi\frac{\partial^2\chi}{\partial Y^2}\right\} + \cdots \end{aligned} \tag{4.89}$$

In the incompressible case the conventional method is to introduce *one* independent variable $\eta \propto y/x^q$, and to try the unknown stream function in the form $\Psi \propto x^p f(\eta)$. This procedure works also in the more general case $U_\infty = \beta x^m$ and leads to $p = 1$, $q = 0$, if $m = 1$. In analogy to the incompressible case we put

$$\eta = \left(\frac{\beta}{\nu_S}\right)^{1/2} Y, \qquad \chi = (\nu_S\beta)^{1/2} x\left[f_1(\eta) + \frac{\beta^2 x^2}{a_S^2} f_3(\eta) + \cdots\right] \tag{4.90}$$

(Note that, according to (4.89), only odd terms in x are required.)

Substitution of (4.90) in the equation of motion (4.83) and comparison of the coefficients of x, x^3, etc., leads to a set of differential equations for the f_n. We quote the first two of them:

$$\begin{aligned} & f_1'^2 + f_1 f_1'' = 1 + f_1''' \\ & 4f_1'f_3' - 3f_1''f_3 - f_1 f_3'' = \frac{\gamma - 1}{2}(1 - f'^2) - \frac{\gamma}{2} f_1 f_1'' + f_3''' \end{aligned} \tag{4.91}$$

. . . .

The first equation is indeed that which belongs to the incompressible problem $G = 1$; it is a special case of the *Falkner-Skan equation*[23] associated with $U_\infty = \beta x^m$, and was already given by Blasius in 1908.

Each of the successive equations in the set (4.91) introduces one new function $f(\eta)$, so that each equation is for one unknown only once the preceding equations have been solved, but note that each equation represents a two-point boundary value problem: for $\eta = 0$ all f_n and f_n' vanish [cf. equations (4.49) and (4.50)], for $\eta = \infty$ there is $f_1' = 1$ and $f_3' = f_5' = \cdots = 0$.

A tabulation of f_3 is found in (17), including its first and second derivatives. It turns out that $f_3'(\eta)$ is confined to the range ± 0.01,

[23] Cf. (1), p. 141.

and since this is the first term that modifies the associated "incompressible" χ_Y-value $\beta x f_1'(\eta)$, one concludes that the deviation from that value is of the order of 1% up to $x \sim a_S/\beta$. This means that the Mach number dependence of the U-distribution is essentially contained in the change of scale from Y to y. To evaluate U one must develop the inverse of equation (4.54), that is

$$y = \left(\frac{p_S}{p}\right)^{1/2} \int_0^Y \frac{T}{T_S}\,dY = \left(1 - \frac{\gamma - 1}{2}\frac{\beta^2 x^2}{a_S^2}\right)^{\gamma/2(\gamma-1)} \int_0^Y \frac{T_S}{T}\,dY \quad (4.92)$$

Here T/T_S has to be expressed in terms of U, using the integral of the energy equation, (3.9′), which we may write in the form $\frac{T}{T_S} = 1 - \frac{\gamma - 1}{2a_S^2} U^2$. This permits an expansion of (4.92) in terms of integrals over $f_1'^2$, etc., the leading term of which is given by

$$y \sim \left(\frac{p_S}{p}\right)^{1/2} \left(\frac{\nu_S}{\beta}\right)^{1/2} \left[\eta - \frac{\gamma - 1}{4a_0^2}\beta^2 x^2 \left(\eta + f_1'' + f_1 f_1' - f_1''(0)\right)\right] \quad (4.93)$$

In (17) Howarth also studies the influence of compressibility on the known incompressible solution for the more general flow near a stagnation point, $U_\infty = \sum_1 \beta_n x^n$, and for the retarded flow, $U_\infty = U_0 - U_1 x$. The general assumptions are the same as before, the method in the first problem is an expansion similar to (4.90)

$$\chi = (\nu_S \beta_1)^{1/2} x [F_1(\eta) + x F_2(\eta) + x^2 F_3(\eta) \cdots] \quad (4.94)$$

but the aim is to obtain a subsidiary system of differential equations that is independent of the velocity coefficients β_n and the parameter a_S. This becomes possible by splitting up the functions $F_n(\eta)$ as follows:

$$F_1 = f_1, \quad F_2 = (3\beta_2/\beta_1) f_2, \quad F_3 = (4\beta_3/\beta_1) g_3 + (4\beta_2^2/\beta_1^2) h_3 + (\beta_1^2/a_S^2) k_3,$$

etc. The first few of these functions (but not k_3) occur already in Howarth's paper[24] of 1934, but the convergence is slow and the number of new functions to be introduced grows rapidly with increasing n.

The principal aim of these developments is to apply subsidiary functions obtained in certain incompressible cases to estimate the compressibility effect in a "simplified" compressible fluid.

[24] The retarded incompressible flow problem is treated in Howarth's paper of 1938 (*Proc. Roy. Soc. (London)*, A164, 547, part I), the accelerated incompressible flow is found in the same author's ARS Rep. & Mem. No. 1632 (1934).

B. The Dependent Variable $1/\rho$; The General Problem of Series Expansion (Cope and Hartree). An ambitious effort along similar lines is found in Cope and Hartree's paper (18), the main part of which is devoted to the series solution of the problem $\sigma \neq 1$, $\mu \propto T^{\omega}$, p_x = constant, $(T_y)_w = 0$. This paper has a more general significance in that it discusses in detail a number of interesting questions that arise in the preparation of the problem for an automatic digital computer, in this case the ENIAC at Aberdeen, Md.

For these details the reader is referred to (18), a review of which is not intended here. The basic steps, however, will be presented, partly also because they include the form of the boundary layer equations used by Emmons and Brainerd in (14,15).

We start from the form (2.20b) of the energy equation and introduce the variable $1/\rho$ instead of T; the second term can be transformed, when (2.19) is used, into

$$p\rho \operatorname{div} \mathbf{q} = -p\mathbf{q} \cdot \operatorname{grad} \rho = \rho^2 p\mathbf{q} \cdot \operatorname{grad} (1/\rho)$$

Then the first member of (2.20b) takes the form

$$(c_v/R)\rho p\mathbf{q} \cdot \operatorname{grad} (1/\rho) + \rho p\mathbf{q} \cdot \operatorname{grad} (1/\rho) + (c_v/R)Up_x$$

and the whole equation of energy may be written, after division by $p[c_v/R + 1] = p\gamma/(\gamma - 1)$

$$\rho\left(U\frac{\partial}{\partial x} + V\frac{\partial}{\partial y}\right)\frac{1}{\rho} + \frac{U}{\gamma}\frac{p_x}{p} = \frac{\partial}{\partial y}\left[\frac{\mu}{\sigma}\frac{\partial}{\partial y}\left(\frac{1}{\rho}\right)\right] + \frac{\gamma - 1}{\gamma}\frac{\mu}{p}\left(\frac{\partial U}{\partial y}\right)^2 \tag{4.95}$$

In the "restricted" problem, U, $V\sqrt{x}$, and ρ are functions of $y/\sqrt{x}$. This suggests also in the present case the introduction of $y/\sqrt{x}$ instead of y (x being the other independent variable). At the same time $V\sqrt{x}$ will be introduced instead of V. By reference to a standard state s, a *velocity of approach* U_s and a standard velocity gradient $U_s' > 0$, we define the following[25] dimensionless variables ξ, η_1:

$$x = \frac{\xi U_s}{U_s'}, \qquad y = 2\eta_1\left(\frac{U_s}{\nu_s x}\right)^{-1/2} \tag{4.96}$$

The dimensionless velocities u and v are defined by

$$U = uU_s, \qquad v = V\left(\frac{x}{U_s\nu_s}\right)^{1/2} \tag{4.97}$$

[25] The factor 2 has already been introduced by Blasius (see footnote to p. 29) and is kept for easier comparison of results, the notation η_1 prepares for a later transformation, but η_1 is identical with the Blasius variable earlier denoted by η. Any function with the subscript ∞ refers to the point (x, ∞) as in (4.55′).

The operator q · grad which occurs in (4.95) and in the equation of motion (2.17) transforms as follows:

$$U\frac{\partial}{\partial x} + V\frac{\partial}{\partial y} = \frac{U_s'}{2\xi}\left[2u\xi\frac{\partial}{\partial \xi} - (\eta_1 u - v)\frac{\partial}{\partial \eta_1}\right] \tag{4.98}$$

This suggests the dependent variable

$$w = \frac{\rho}{\rho_s}(\eta_1 u - v) \tag{4.99}$$

instead of v.

The equation of continuity transforms by (4.97) and (4.98) into

$$\left(2\xi\frac{\partial}{\partial \xi} - \eta_1\frac{\partial}{\partial \eta_1}\right)\frac{\rho u}{\rho_s} + \frac{\partial}{\partial \eta_1}\left(\frac{\rho v}{\rho_s}\right) = 0 \tag{4.100}$$

and, further, by (4.99) into

$$\frac{\partial w}{\partial \eta_1} = \left(1 + 2\xi\frac{\partial}{\partial \xi}\right)\frac{\rho u}{\rho_s} \tag{4.101}$$

The dimensionless pressure gradient is written as

$$\frac{\partial}{\partial \xi}\left(\frac{p}{p_s}\right) = \frac{U_s}{U_s'}\frac{p_x}{p_s} = \frac{\rho_s U_s^2}{p_s}\cdot\frac{p_x}{\rho_s U_s U_s'} = \gamma M_s^2 \cdot \bar{C} \tag{4.102}$$

where M_s is the Mach number of approach; the positive dimensionless constant $\bar{C}$ serves as measure number for the actual (constant) pressure gradient p_x. By (4.98) and (4.99) the equation of motion (2.17) transforms into

$$2\frac{\rho}{\rho_s}u\xi\frac{\partial u}{\partial \xi} - w\frac{\partial u}{\partial \eta_1} = -2\xi\bar{C} + \frac{1}{2}\frac{\partial}{\partial \eta_1}\left(\frac{\mu}{\mu_s}\frac{\partial u}{\partial \eta_1}\right) \tag{4.103}$$

and the energy equation (4.95) takes the form

$$\begin{aligned} 2\frac{\rho}{\rho_s}u\xi\frac{\partial}{\partial \xi}\left(\frac{\rho_s}{\rho}\right) - w\frac{\partial}{\partial \eta_1}\left(\frac{\rho_s}{\rho}\right) + 2M_s^2\frac{p_s}{p}u\xi\bar{C} \\ = \frac{1}{2\sigma}\frac{\partial}{\partial \eta_1}\left[\frac{\mu}{\mu_s}\frac{\partial}{\partial \eta_1}\left(\frac{\rho_s}{\rho}\right)\right] + \frac{\gamma - 1}{2}M_s^2\frac{\mu}{\mu_s}\frac{p_s}{p}\left(\frac{\partial u}{\partial \eta_1}\right)^2 \end{aligned} \tag{4.104}$$

The second derivatives in (4.103) and (4.104) suggest as before [e.g., in section IV, 2A equation (4.62)] a distortion of the y-scale. Here we put

$$\eta = \int_0^{\eta} \frac{\mu_s}{\mu}\,d\eta_1 \tag{4.105}$$

The operator q · grad which occurs in (4.95) and in the equation of motion (2.17) transforms as follows:

$$U\frac{\partial}{\partial x} + V\frac{\partial}{\partial y} = \frac{U_s'}{2\xi}\left[2u\xi\frac{\partial}{\partial \xi} - (\eta_1 u - v)\frac{\partial}{\partial \eta_1}\right] \tag{4.98}$$

This suggests the dependent variable

$$w = \frac{\rho}{\rho_s}(\eta_1 u - v) \tag{4.99}$$

instead of v.

The equation of continuity transforms by (4.97) and (4.98) into

$$\left(2\xi\frac{\partial}{\partial \xi} - \eta_1\frac{\partial}{\partial \eta_1}\right)\frac{\rho u}{\rho_s} + \frac{\partial}{\partial \eta_1}\left(\frac{\rho v}{\rho_s}\right) = 0 \tag{4.100}$$

and, further, by (4.99) into

$$\frac{\partial w}{\partial \eta_1} = \left(1 + 2\xi\frac{\partial}{\partial \xi}\right)\frac{\rho u}{\rho_s} \tag{4.101}$$

The dimensionless pressure gradient is written as

$$\frac{\partial}{\partial \xi}\left(\frac{p}{p_s}\right) = \frac{U_s}{U_s'}\frac{p_x}{p_s} = \frac{\rho_s U_s^2}{p_s}\cdot\frac{p_x}{\rho_s U_s U_s'} = \gamma M_s^2 \cdot \bar{C} \tag{4.102}$$

where M_s is the Mach number of approach; the positive dimensionless constant $\bar{C}$ serves as measure number for the actual (constant) pressure gradient p_x. By (4.98) and (4.99) the equation of motion (2.17) transforms into

$$2\frac{\rho}{\rho_s}u\xi\frac{\partial u}{\partial \xi} - w\frac{\partial u}{\partial \eta_1} = -2\xi\bar{C} + \frac{1}{2}\frac{\partial}{\partial \eta_1}\left(\frac{\mu}{\mu_s}\frac{\partial u}{\partial \eta_1}\right) \tag{4.103}$$

and the energy equation (4.95) takes the form

$$2\frac{\rho}{\rho_s}u\xi\frac{\partial}{\partial \xi}\left(\frac{\rho_s}{\rho}\right) - w\frac{\partial}{\partial \eta_1}\left(\frac{\rho_s}{\rho}\right) + 2M_s^2\frac{p_s}{p}u\xi\bar{C}$$
$$= \frac{1}{2\sigma}\frac{\partial}{\partial \eta_1}\left[\frac{\mu}{\mu_s}\frac{\partial}{\partial \eta_1}\left(\frac{\rho_s}{\rho}\right)\right] + \frac{\gamma - 1}{2}M_s^2\frac{\mu}{\mu_s}\frac{p_s}{p}\left(\frac{\partial u}{\partial \eta_1}\right)^2 \tag{4.104}$$

The second derivatives in (4.103) and (4.104) suggest as before [e.g., in section IV, 2A equation (4.62)] a distortion of the y-scale. Here we put

$$\eta = \int_0^{\eta}\frac{\mu_s}{\mu}\,d\eta_1 \tag{4.105}$$

variable η_1, by transformation (4.105), greatly facilitates the computational work by absorbing the principal influence of M_s on the zero-order functions, which come out rather insensitive to a change of M_s.

C. Numerical Solution of the Restricted Problem (Emmons and Brainerd)—Asymptotic Integration (Meksyn). The zero-order functions are, of course, directly related to the solution of the corresponding boundary layer problem with *vanishing pressure gradient.* In the latter case the expansions (4.107) reduce to the first terms, and the connection with the original variables is given by

$$h_0 = 2u, \qquad f_0 = 2w = 2\frac{\rho}{\rho_s}(\eta_1 u - v), \qquad r_0 = r = \frac{T/T_s - 1}{M_s^2} \tag{4.110}$$

This problem $[\sigma \neq 1,\ \mu \propto T^\omega,\ p_x = 0,\ (T_y)_w = 0]$ was for the first time systematically investigated in 1941 and 1942 by Emmons and Brainerd in (14,15). In their first paper constant viscosity is assumed; in the second paper this assumption is dropped.[26]

The authors use the boundary layer equations in the form (4.101), (4.103), (4.104), but for $p_x = 0$. This implies not only $\bar{C} = 0$, but the equations, as pointed out earlier (cf. p. 60) become altogether independent of ξ. (In the formal way, the new equations can be obtained by going to the limit $U_s' \to 0$.) Adopting the notations[27] φ, η, $\underline{\xi}$, and θ of (15), we have the relations

$$\varphi = \frac{\mu}{\mu_s}, \qquad \theta = \frac{T}{T_s} = \frac{\rho_s}{\rho} \quad (\text{since } p = \text{constant}), \qquad \eta = 2\eta_1 \tag{4.111}$$
$$\underline{\xi} = \frac{2w\rho_s}{\rho} = 2(\eta_1 u - v) = \eta u - 2v$$

The standard subscript s may be identified with the subscript ∞, then $p/p_s = 1$ and the modified equations (4.101), (4.103), (4.104) read, when prime denotes differentiation with respect to η,

$$\frac{d}{d\eta}\left(\frac{\underline{\xi}}{\theta}\right) = \frac{u}{\theta} \quad \text{or} \quad \theta(\underline{\xi}' - u) = \underline{\xi}\theta \tag{4.112}$$

$$-\underline{\xi}u' = 2\theta\frac{d}{d\eta}(\varphi u') \quad \text{or} \quad -\underline{\xi} = 2\varphi\theta\frac{d}{d\eta}\log(\varphi u') \tag{4.113}$$

$$-\frac{\underline{\xi}}{\theta}\theta' = \frac{2}{\sigma}\frac{d}{d\eta}(\varphi\theta') + 2(\gamma - 1)M^2\varphi u'^2 \tag{4.114'}$$

[26] Since σ = constant is adhered to, it follows that k varies with T in the same way as μ does, cf. the definition of σ on p. 26.

[27] But $\underline{\xi}$ instead of ξ to avoid confusion; $\underline{\xi}$ is now a *dependent* variable, η is not to be confused with the η of the preceding section 3B; here it denotes *twice* the Blasius variable; we write θ instead of t in this section to facilitate comparison with the original papers.

or, by (4.112), with the abbreviation

$$b = (\gamma - 1)M^2$$

$$\frac{2}{\sigma}\frac{d}{d\eta}(\varphi\theta') + \underline{\xi}' - u + 2b\varphi u'^2 = 0 \qquad (4.114)$$

These are the basic equations of (15), and those of (14) are obtained by putting $\varphi = 1$. The boundary conditions in the present notation are

$$u,\ \underline{\xi},\ \theta' = 0 \quad \text{at } \eta = 0$$
$$u,\ \theta = 1 \quad \text{at } \eta = \infty$$

One can eliminate $\underline{\xi}$ from (4.114′) [substitution for the left member of (4.114′) according to (4.113)] and obtain

$$\theta'' + \theta'\left[(1 - \sigma)\frac{d \log \varphi}{d\eta} - \sigma\frac{u''}{u'}\right] + \sigma b u'^2 = 0 \qquad (4.114a)$$

If the dependence of φ on T is disregarded, this equation can be integrated when a suitable u-profile is assumed. Equation (4.114a) has been used in this way first by Pohlhausen and later by Eckert and Drewitz in 1940 (22), to estimate the influence of compressibility on the heat transfer[28] for prescribed T_w. For convenience equation (4.114a) is here rewritten in the usual way, that is, with $\varphi = 1$:

$$\theta'' - \sigma\theta' u''/u' + \sigma(\gamma - 1)M^2 u'^2 = 0. \qquad (4.114b)$$

Note that (4.114a) and (4.114b) remains unchanged if η is replaced by $k\eta$, where k is some constant.

The integration of the system (4.112)–(4.114) has been carried out in (14,15) by means of a differential analyzer. The two-point boundary conditions are satisfied by running trial solutions for chosen initial values $(u')_w$ and θ_w. These values are adjusted until the conditions for large η appear to be fulfilled with the desired accuracy. In (14) the case $M = 0$ offers the opportunity to check results against the Blasius solution. On the basis of such a check systematic errors up to 0.3% seem possible, as the authors point out.

Numerical results are presented in the case $\varphi = 1$ for the quantities $(u')_w$, θ_w, $\Theta_w = (T_w - T_\infty)/(T_\infty{}^* - T_\infty) \equiv (\theta_w - 1)/(b/2)$ and $v_\infty = \frac{1}{2}\lim_{\eta\to\infty}(\eta - \underline{\xi})$. The physical V_∞ is related to v_∞ through

[28] It will be noted that the condition $(T_y)_w = 0$ has neither been used in establishing the set (4.101), (4.103), (4.104), nor in the transition to the set (4.112) – (4.114), but the validity of the latter set requires $T_w =$ constant, since the boundary conditions must not depend on ξ.

$$\frac{V_\infty}{U_\infty} = \frac{v_\infty}{\sqrt{\mathrm{Re}(x/L)}} \tag{4.115}$$

The function $\Theta = (\theta - 1)/(b/2)$ is the temperature excess over the free stream temperature measured in units of $T_\infty^* - T_\infty = U_\infty^2/2c_p$.

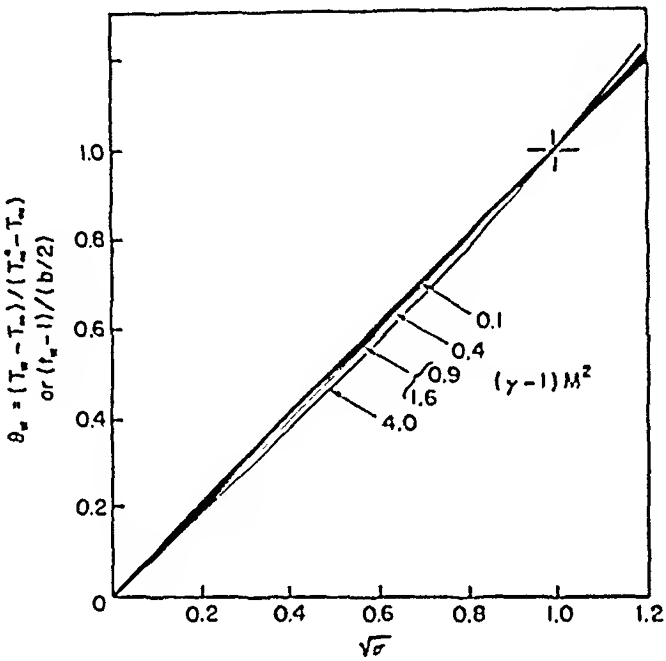


Fig. 4. Θ_W as a function of Prandtl number σ and compressibility number $b = (\gamma - 1)M^2$; μ is constant.

In the simple problem without heat transfer we had (3.9′) which reads in terms of Θ

$$\Theta = 1 - u^2, \quad [\sigma = 1], \tag{4.116}$$

hence, in that case, $\Theta_w = 1$. The tabulated results of Emmons and Brainerd indicate (see Fig. 4) that

$$\Theta_w \sim \sqrt{\sigma} \tag{4.117}$$

is a good approximation in the range tested, namely $M \leqq 3.16$, corresponding to $b = 4$ if $\gamma = 1.4$, and $\sigma = 0, 0.25, 0.733, 1.00, 1.20$.

In (15), the computations are repeated for the viscosity power law with exponent $\omega = 0.768$. Also in this case equation (4.117) seems well enough satisfied, but it is, of course, trivially correct for $\sigma = 0$ and 1, and since the station $\sigma = 0.25$ has been left out in the calculations the deviation midway between $\sqrt{\sigma} = 0$ and $\sqrt{\sigma} = 1$ is not exhibited.

Relation (4.117) appears for the first time in (5) where the excess reading of the plate thermometer in an incompressible fluid with constant

μ was calculated (cf. p. 30). This means solving (4.114b) with $u = \zeta'/2$ and the boundary conditions $\theta_w' = 0$, $\theta_\infty = 1$. But Pohlhausen actually tabulated his result

$$\Theta_w = \tfrac{1}{2}\sigma \int_0^\infty \left[\exp\left(-\sigma \int_0^\eta \zeta d\eta_2\right) \cdot \int_0^\eta \left\{\zeta''^2 \exp\left(\sigma \int_0^{\eta_1} \zeta d\eta_2\right)\right\} d\eta_1\right] d\eta \tag{4.117'}$$

for $0.6 < \sigma < 15$ without mentioning the approximation $\Theta_w \sim \sqrt{\sigma}$, which is excellent in the range $0.5 < \sigma < 2$, characteristic for gases. Since $\zeta = -\zeta'''/\zeta''$, $-\int_0^\eta \zeta d\eta = \log(\zeta''/\alpha)$ and (4.117') may be simplified to

$$\Theta_w = \tfrac{1}{2}\sigma \int^\infty (\zeta'')^\sigma \cdot \int_0^\eta (\zeta'')^{2-\sigma} d\eta_1 d\eta \tag{4.117''}$$

The form (4.117) of the recovery factor[29] Θ_w appears for example, in (1), p. 630, and in (22), Fig. 2, but was undoubtedly known much earlier.[30]

Equation (4.117') is obtained by integrating (4.114b) under the assumption $u = \zeta'/2$, as already pointed out. However, relation (4.117) or its equivalent

$$T_w - T_\infty = \left(\frac{U_\infty^2}{2c_p}\right)\sqrt{\sigma}$$

has a much wider range of validity. As will be seen in Section V

$$T_w - T = \left(\frac{U^2}{2c_p}\right)\sqrt{\sigma} \tag{4.117a}$$

[29] The following terminology is widely used: $T_w - T_\infty$ is called the temperature recovery ΔT; in the case $\sigma = 1$, $\Delta T = \frac{1}{2}(\gamma - 1)M^2$ is denoted by ΔT_{ad} (adiabatic). $\Delta T/\Delta T_{ad}$ is then called the recovery factor, which is approximately $\sqrt{\sigma}$ in the present case.

[30] In order to show that $\Theta_w \sim \sqrt{\sigma}$ for $\sigma \sim 1$, one may put $\sigma = 1 + \epsilon$ in (4.117'') and transform the "inner" integral by parts, writing the integrand in the form $(\zeta'')^{-\epsilon} d\zeta'$. Using $\int_0^\infty \zeta''\zeta' d\eta = 2$, one has

$$2\Theta/\sigma = 2 - \epsilon C(\epsilon)$$

where

$$C(\epsilon) = \int_0^\infty (\zeta'')^{1+\epsilon} \cdot \int_0^\eta \zeta\zeta'(\zeta'')^{-\epsilon} d\eta_1 d\eta$$

The first order term in ϵ is thus obtainable by evaluating $C(0)$. One finds readily

$$C(0) = \tfrac{1}{2}\int_0^\infty \zeta''\zeta^2 d\eta = -\tfrac{1}{2}\int_0^\infty \zeta\zeta''' d\eta = 1$$

on integration by parts. Hence

$$\Theta_w(\sigma) \sim \sigma(1 - \tfrac{1}{2}\epsilon)$$

which equals $\sqrt{\sigma}$ in the same order of approximation.

represents quite generally a good approximate integral of the energy equation when $(T_y)_w = 0$ [cf. (18), p. 9 where the form $c_pT + \frac{1}{2}\sqrt{\sigma}\,(U^2 + V^2) =$ constant is suggested].

We shall come back to Emmons and Brainerd's contributions in the section on results and close this section with a brief review of a method of integration of the system (4.112)–(4.114) that leads to numerical results of surprisingly high accuracy in view of the small amount of computational work involved. The method was proposed by D. Meksyn in a paper that deals with the integration of the equation of Falkner and Skan (23) and applied by the same author in (24) to the system under consideration.

Let us first take the Blasius equation as example, using *the independent variable of this section* η. On denoting differentiation with respect to η by primes, the Blasius equation reads

$$2\zeta''' + \zeta\zeta'' = 0, \qquad \zeta(0) = \zeta'(0) = 0, \qquad \zeta'(\infty) = 1 \tag{4.118}$$

since $u = \frac{1}{2}(d\zeta/d\eta_1) = \zeta'$. For large η, $\zeta \sim \eta +$ constant, hence, on putting $\zeta = \eta + \bar{\zeta}$ and neglecting the quadratic term one has approximately for large η

$$2\bar{\zeta}''' + \eta\bar{\zeta}'' = 0, \qquad \bar{\zeta}'' = ce^{-\eta^2/4} \tag{4.119}$$

from which the asymptotic behavior of ζ'' *follows:*

$$\zeta'' = O(e^{-\eta^2/4}) \quad \text{for large } \eta \tag{4.120}$$

This estimate has been known all along and is the starting point of Blasius original integration method.

Now, according to Meksyn's approach, it seems rather a disadvantage from the point of view of the numerical calculation to exhibit in the trial solution the *correct* asymptotic behavior of ζ. For the purpose in question, which is the quick determination of u_w' or $\zeta''(0)$, a representation of the form

$$\zeta'' = A(\eta)e^{-B(\eta)} \tag{4.121}$$

appears desirable. But since the function $B(\eta)$ impresses itself on $\zeta' = \int_0^\eta \zeta''\,d\eta$ only through its values in the neighborhood of 0, $B(\eta)$ should be determined by satisfying (4.118) as well as possible for *small* η. Now

$$\zeta' \sim \frac{\beta\eta^2}{2} \tag{4.122}$$

is the first term of the expansion at 0, and, on linearizing (4.118) with $\zeta \sim \beta\eta^3\!/\!_6$, one obtains for ζ''

$$\zeta'' \sim Ae^{-\beta\eta^3/12} \tag{4.123}$$

where $A \equiv \beta$ to be consistent with (4.122). The constant β can now be calculated according to the condition $\zeta'(\infty) = 1$, from

$$\int_0^\infty \beta e^{-\beta\eta^3/12} d\eta = 1 \tag{4.124}$$

and provides a first approximation of the Blasius constant $\alpha = 4\zeta''(0) \sim 4\beta$. One obtains from (4.124)

$$\alpha \sim \frac{6}{[(-\frac{2}{3})!]^{3/2}} = 1.346$$

which deviates by 1.5% from the accepted value.

This idea can be extended to solve the system (4.112)–(4.114). In doing so one approximates those functions that approach zero asymptotically by expressions of the form $A(\eta) \exp[-B(\eta)]$ and tries to determine $B(\eta)$ from a linearized equation. It is important to obtain a $B(\eta)$ such that the term of highest degree in η is positive, which can be effectuated by cutting expansion of the type (4.122) at the proper place.

In view of equation (4.113) which represents a linear homogeneous equation of first order for u' when we restrict ourselves to the case $\varphi = 1$, we start with the linearization of that equation. The initial terms of the expansions are easily found to be

$$u = \beta\eta, \qquad \theta = \theta_w, \qquad \xi = \frac{\beta\eta^2}{2}; \quad \beta, \theta_w > 0 \tag{4.125}$$

Substitution in (4.113) yields

$$u'' + \frac{\beta\eta^2}{4\theta_w} u' = 0 \tag{4.126a}$$

hence

$$u' = \beta \exp\left(-\frac{\beta}{12\theta_w}\eta^3\right) \tag{4.126b}$$

This approximative result for u' satisfies the sign condition for the leading term; and $A(\eta) = \beta$ makes it consistent with the first relation (4.125). The condition $u(\infty) = 1$ applied to (4.126b) furnishes an equation like (4.124), but this time for β *and* θ_w.

When we substitute in (4.114b) for u' according to (4.126b), we obtain a first order inhomogeneous equation for θ'. Now the function θ' which is again of the asymptotically vanishing type, should be approximated just by such a solution of a linearized first order equation, and the linearization consists here simply in the indicated substitution for u'. One then obtains

$$\theta' = -b\beta^2\sigma\left(\int_0^\eta \exp\left[-\frac{(2-\sigma)\beta}{12\theta_w}\eta^3\right] d\eta\right) \cdot \exp\left[-\frac{\sigma\beta}{12\theta_w}\eta^3\right] \tag{4.127}$$

which exhibits the form $A(\eta) \cdot \exp[-B(\eta)]$ and satisfies $\theta'(0) = 0$. The function θ is found by integrating (4.127) and adjusting the integration constant so that $\theta(\infty) = 1$, whence

$$\theta = 1 + \int_{\infty}^{\eta} \theta' d\eta \tag{4.128}$$

where the right member depends on the parameters β and θ_w. Now we can set up the second condition for β and θ_w in the form

$$\theta_w = 1 + \int_{\infty}^{0} \theta' d\eta$$

the right member being a double integral requiring evaluation by numerical methods. We thus have two conditions to determine β and θ_w for given Mach and Prandtl numbers. For $\sigma = 0.733$ Meksyn obtains

$$\theta_w = 1 + 0.424b, \quad \beta^2\theta_w = 0.117 \tag{4.129}$$

The determination of $\underline{\xi}$ follows similar lines. In this case $\underline{\xi}' = u + \eta u' - 2v'$ by (4.111), and $\underline{\xi}' - u$ therefore vanishes asymptotically. This function is obtained from (4.112) and (4.113) in the required form

$$\underline{\xi}' - u = \frac{2u''}{u'}\theta' \quad \text{or} \quad \underline{\xi}' - u = -\frac{2}{\sigma}\theta'' - 2bu'^2 \tag{4.130}$$

by (4.114), this being the most convenient way to obtain v by integration of $\underline{\xi}' - u$.

The numerical results for u_w', θ_w, and v_∞ agree very well with the tabulated results of (14). Only the case $b = 4$ required a more elaborate start; one had to substitute the first *two* terms of the expansion of $\underline{\xi}/(2\theta)$ in (4.113) to obtain the same accuracy as before.

Meksyn treats in the same paper (24) the problem $\varphi \neq 1$ with equal success, but it should be pointed out that in both cases the agreement of the two sets of results is only shown for the *boundary values* of the functions u, v, θ. They doubtlessly are the values of foremost practical interest, but one would like to know how good the overall accuracy of the method is.

V. Results

This section gives a survey of the answers so far obtained for a number of practically important questions concerning the laminar boundary layer in a compressible fluid. For numerical results the reader is referred in each case to the original paper, but characteristic diagrams will be reproduced and the method of analysis indicated in connection with the preceding section IV.

1. The Simple Problem

The problem [$\sigma = 1$; $p_x = 0$; $T_w =$ constant] is the principal subject of (8,11). It is treated in Crocco's earlier papers and appears as a special case in the very complete diagrams and tables of (16). A general idea of the u- and t-distribution is obtained from our Figs. 1 and 2, which have been taken from (11).

The simple problem offers a good opportunity to study the effect of the variation of one of the parameters. The product $C_{D_f}\sqrt{\text{Re}}$ or $2K_0$ for which the asymptotic value (3.2) was obtained by Busemann

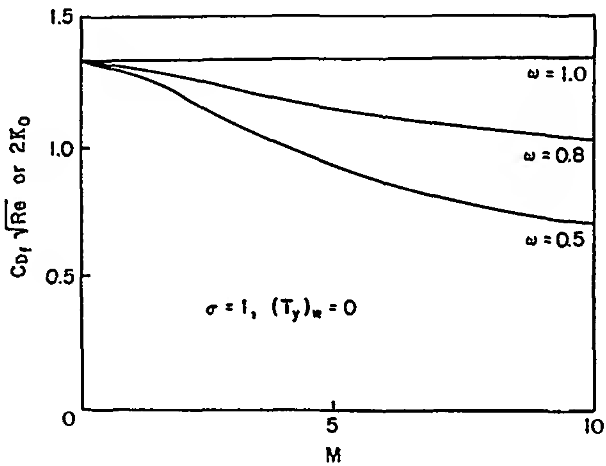


FIG. 5. The coefficient of friction $C_{d_f}\sqrt{\text{Re}}$ for vanishing heat transfer.

has been investigated in its dependence on ω in (13) (Hantzsche and Wendt). Figure 5 shows the results in the case $(t_y)_w = 0$. For $\omega = 1$, this product is constant [cf. the paragraph following equation (3.20)]; in appraising the effect of a specified $\omega \neq 1$ it should be remembered that in (13) and other papers of the same authors the standard temperature in the viscosity law is T_w [cf. equation (4.36)]. The quantity corresponding to K_0 in Hantzsche and Wendt's analysis is $g_0 \equiv g(0)$ [see equation (4.41)], represented in Fig. 6, again for the case of vanishing heat transfer. This diagram shows that g_0, which is essentially $K_0(1 + b/2)^{(1-\omega)/2}$, is somewhat less sensitive to changes in the parameters M and ω, than K_0 itself.

In the more general case of $(t_y)_w \neq 0$, g_0 is a function of t_w. It is represented in Fig. 7 for $M = 5$ and $\omega = 0.8$. (When $\omega = 1$, g_0 equals $\alpha/\sqrt{8}$ independently of t_w, cf. the broken line in Fig. 7.) The physical interpretation of g_0 can be based either on its connection with the stress

τ_w or, more directly, on its thermodynamical meaning. From Reynolds's analogy (3.14) and from (4.37) one finds the local heat transfer coefficient β^*_{loc} referred to the excess of the equilibrium temperature T_e which in

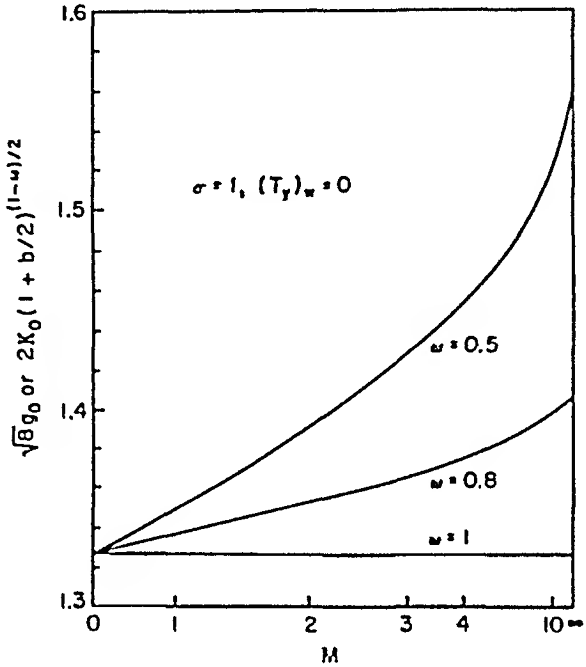


FIG. 6. $\sqrt{8}\,g_0$ versus M for different values of ω.

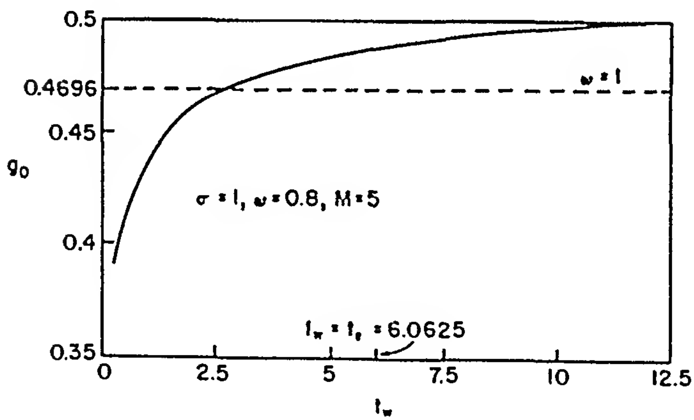


FIG. 7. g_0 versus t_w for constant M, at the same time a plot of $\beta^*_{loc}\sqrt{x}$ versus t_w.

our case equals T^*_∞, over T_∞:

$$\beta^*_{loc} \equiv \frac{q_w}{T_w - T^*_\infty} = c_p \sqrt{\frac{U_\infty \rho_w \mu_w}{2x}}\, g_0 \tag{5.1}$$

Thus $\beta^*_{loc}\sqrt{x} \propto g_0$ and Fig. 7 may be interpreted as a graph of $\beta^*_{loc}\sqrt{x}$ if the ordinate scale is adjusted correspondingly.

In the case under consideration it is also possible to estimate the thickening of the boundary layer due to compressibility. Howarth does that in (17) for the viscosity exponent $\omega = 1$. Then the Blasius function $\zeta(\eta)$ solves the problem in terms of

$$\eta = \frac{1}{2}\left(\frac{U_\infty}{\nu_\infty x}\right)^{\frac{1}{2}} Y \tag{5.2}$$

[cf. equation (4.53) with $dU_\infty/dx = 0$ and equation (4.6a) with $p_x = 0$ and ρ = constant.] Howarth defines now the ordinate y_i of the associated incompressible flow by

$$\eta = \frac{1}{2}\left(\frac{U_\infty}{\nu_\infty x}\right)^{\frac{1}{2}} y_i \tag{5.3}$$

On expressing Y in (5.2) in terms of the physical coordinate y_c of the compressible flow [cf. equation (4.54)], one obtains for not too small η the relation

$$\frac{y_c - y_i}{y_i} \sim 0.60\,\frac{\gamma - 1}{\eta} M^2 \tag{5.4}$$

giving the relative difference of ordinates of those points (with the same x) in the two flows, where the stream function and the other functions of η have the same value. If $\eta = 3$ is taken as the definition of the boundary layer thickness δ, corresponding to $u = 0.999$, then

$$\frac{\delta_c - \delta_i}{\delta_i} = 0.08M^2 \quad \text{for } \gamma = 1.4 \tag{5.5}$$

2. The Restricted Problem for $\omega = 1$

When the assumption $\sigma = 1$ is dropped, the simple integral of the energy equation (3.9) is no longer available. Even the restricted problem [$p_x = 0$; T_w = constant] presents then formidable difficulties to a solution in general terms. There is, however, the particular choice of the viscosity exponent, $\omega = 1$ [equation (4.11)], which very considerably simplifies the problem.

In that case the integral of the *equation of motion* is known beforehand in the form of the Blasius solution. This has already been shown in the discussion following equation (4.14) and applies equally well to the first of Crocco's equations (4.31a). Let $\bar{K}(u)$ denote the solution of (4.31a) if $\omega = 1$ (or $\bar{\mu}\bar{\rho} = 1$); as can be easily verified, $\bar{K}(u)$ is related to the Blasius function ζ through

$$\zeta = -\frac{d\bar{K}}{du} \tag{5.6}$$

(ζ being considered as a function of $u = \frac{1}{2}d\zeta/d\eta$). This function is plotted in Fig. 8a [Figs. 8–11 have been taken from (16)].

Substitution of $\bar{K}(u)$ in (4.31b) yields a linear first order equation for t' which can be solved by quadratures.[31] The solution can be given the

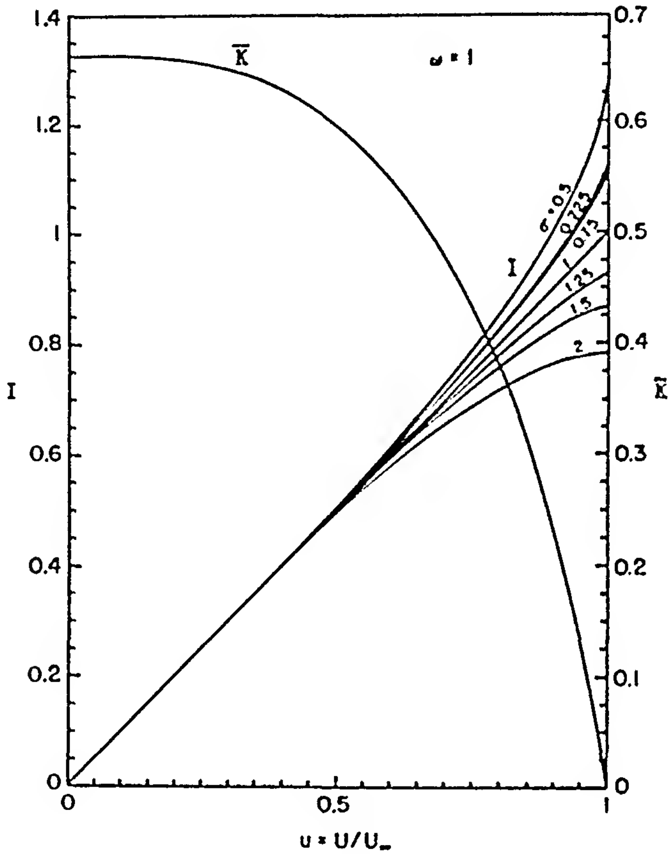


FIG. 8a. Crocco's stress function $\bar{K}(u)$ and the functions $I(u, \sigma)$ for varying Prandtl number.

form of the Crocco integral (3.13):

$$t(u) = t_w - (t_w - 1)\theta^{\mathrm{I}}(u) + (\gamma - 1)M^2\theta^{\mathrm{II}}(u) \tag{5.7}$$

where θ^{I} and θ^{II} are functions of u that depend on σ and have the general character

$$\theta^{\mathrm{I}}(u,\sigma) \sim u, \qquad \theta^{\mathrm{II}}(u,\sigma) \sim \tfrac{1}{2}u(1 - u) \tag{5.8}$$

when $\sigma \sim 1$ (for $\sigma = 1$, the approximations (5.8) become correct). To write up the functions θ^{I} and θ^{II} we define first

[31] Note the formal parallelism between the cases $\omega = 1$ and $\omega = 0$, comparing (4.31b) with (4.114b).

$$I(u,\sigma) = \int_0^u \left(\frac{\bar{K}}{\bar{K}_0}\right)^{\sigma-1} du_1 \tag{5.9a}$$

$$J(u,\sigma) = \int_0^u \left(\frac{\bar{K}}{\bar{K}_0}\right)^{\sigma-1} du_1 \int_0^{u_1} \left(\frac{\bar{K}}{\bar{K}_0}\right)^{1-\sigma} du_2 \tag{5.9b}$$

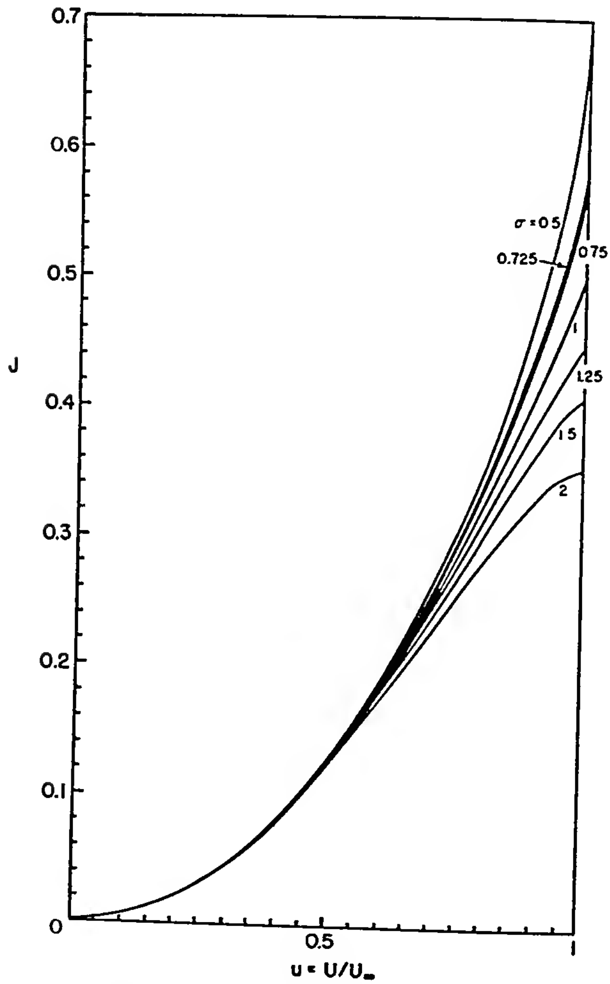


FIG. 8b. The functions $J(u, \sigma)$ for varying Prandtl number.

The functions[32] $I(u)$ are shown in Fig. 8a. Compared with $I(u)$, the functions $J(u)$ start at $u = 0$ in a parabolic manner and reach about half the end value of the corresponding $I(u)$, cf. Fig. 8b. The functions introduced in (5.7) are now given by

[32] The argument σ will be exhibited only in cases where doubts are possible. $\bar{K}_0$ is identical with K_0 introduced in (3.20).

$$\theta^{\mathrm{I}} = I/I(1), \qquad \theta^{\mathrm{II}} = \sigma(\theta^{\mathrm{I}}J(1) - J) \tag{5.10}$$

and represented in Fig. 9.

It is of course possible to give in the restricted problem with $\omega = 1$ an explicit expression for Reynolds' analogy, when one combines (5.7) with (3.22) and (3.14″). One obtains

$$\frac{Q_w}{F_w} = -\frac{k_w T_\infty}{\mu_w U_\infty}\left[\frac{\sigma J(1)}{I(1)}(\gamma - 1)M^2 - \frac{1}{I(1)}(t_w - 1)\right] \tag{5.11}$$

An analogous formula can be developed for the Nusselt number, cf. equations (3.25) and (3.25a).

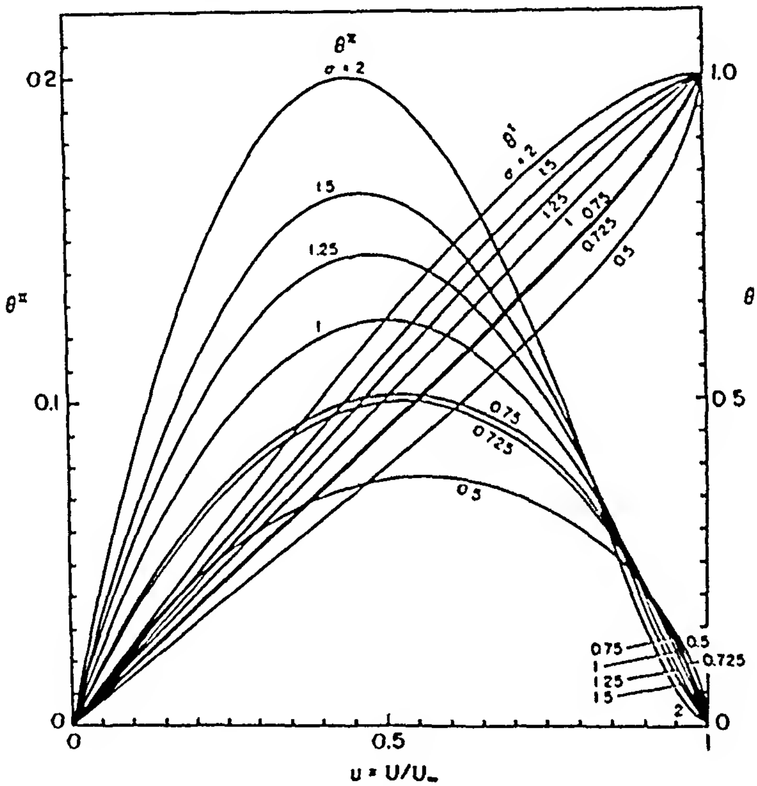


FIG. 9. Crocco's functions $\theta^{\mathrm{I}}(u, \sigma)$ and $\theta^{\mathrm{II}}(u, \sigma)$ for varying Prandtl number.

3. *The Restricted Problem for $\omega \neq 1$*

Also in the case $\omega \neq 1$ Crocco was able to obtain important results of fairly general character, but only on the basis of extensive numerical work. It turns out that *for constant M the slope of the temperature profile at the wall, $(dt/du)_w$, is practically the* same *linear function of t_w,* whatever the chosen viscosity exponent. The computations have been carried out for $\sigma = 0.725$, but there is hardly any doubt that the same statement would hold for any $\sigma \sim 1$. In Fig. 10, the solid straight lines correspond

to $\omega = 1$, in which case the differentiation of (5.7) yields

$$\left(\frac{dt}{du}\right)_w = \frac{1}{I(1)}(1 - t_w + \sigma J(1)(\gamma - 1)M^2) \tag{5.12}$$

The values of $(t_u)_w$ computed for the set $\omega = 0.50, 0.75, 1.25$ and a chosen value of M lie quite close to that straight line (5.12) which belongs to the chosen M-value. These results have already been given by Crocco in

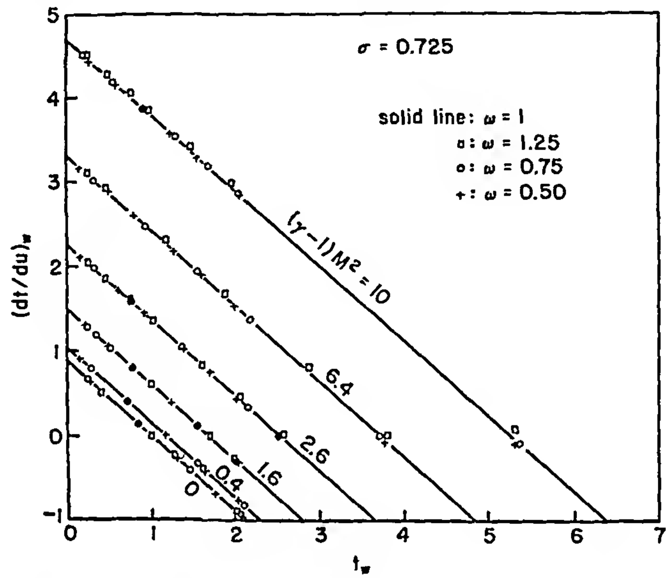


FIG. 10. $(dt/du)_w$ versus t_w for varying ω and M; the solid lines correspond to equation (5.12).

(12) where they were averaged (under disregard of $\omega = 1.25$ for which the deviations were largest), in the following formula

$$\left(\frac{dt}{du}\right)_w = 0.890(1 - t_w) + 0.378(\gamma - 1)M^2 \tag{5.13}$$

The computations were repeated with an improved technique and under adoption of Sutherland's formula[33] instead of the viscosity power law. We have therefore

$$\bar{\rho}\bar{\mu} = \frac{\theta_s^{1/2} + \theta_s^{-1/2}}{(t/\theta_s)^{1/2} + (t/\theta_s)^{-1/2}} \tag{5.14}$$

[33] In Crocco's papers, particularly in (16), the variable T appears everywhere in the form $I = c_pT$ and all assumptions concerning ρ and μ are stated in terms of the enthalpy I. Thus c_p may still depend on the temperature, but σ must, of course, be constant.

and θ_s stands for T_s/T_∞ [cf. equation (4.12) for T_s]. The actual values of θ_s were 0, $\frac{1}{3}$, 1, 3. The corresponding curves (5.14) are tangential at $t = 1$ to the $t^{\omega-1}$ curves previously used to represent the product $\bar{\rho}\bar{\mu}$. (Note that $\theta_s = 1$ corresponds to $\omega = 1$.)

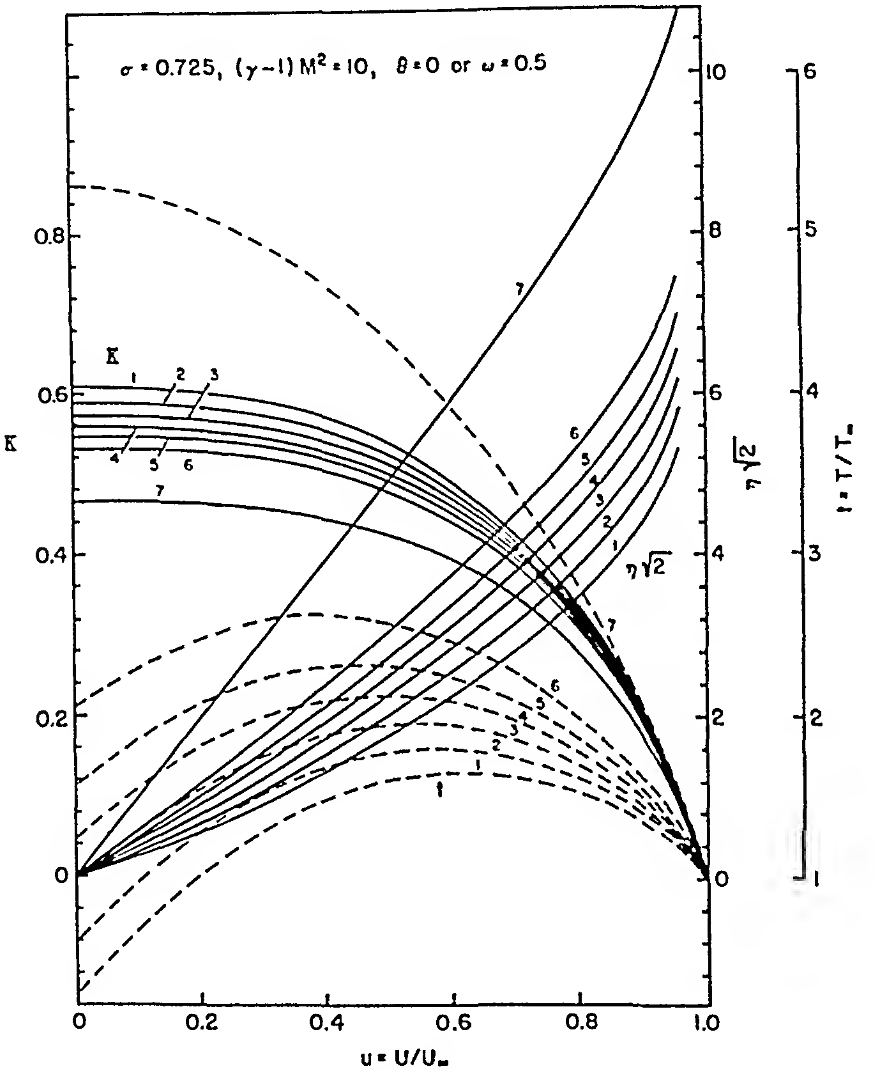


FIG. 11. Stress and temperature fields of the restricted problem for varying boundary conditions $t_w = T_w/T_\infty$.

Calculations of the velocity and temperature fields were carried out for six integer M-values 0, 1, $\cdots$, 5, and in each case six t_w values were selected to cover roughly the interval 0.2–2.0. In Fig. 11 the K and t distributions have been reproduced for $M = 5$; θ_s equals here zero, in which case (5.14) coincides *all along* with $\bar{\rho}\bar{\mu} = t^{-\frac{1}{2}}$, or $\bar{\mu} = t^{\frac{1}{2}}$ (Busemann's assumption in 1935). This extreme case corresponds to

$T_\infty = \infty$ (since T_s is a positive number); for air, it begins to approach reality when the free stream temperature goes beyond a thousand degrees K. The curves labeled 7 represent the case $(t_u)_w = 0$. The physical velocity field is given by the graphs $\eta\sqrt{2}$ vs u, η being the Blasius variable. The Prandtl number σ equals 0.725 throughout.

The $(t_u)_w$ vs t_w-lines for constant M resulting from these computations were again found to be straight regardless of the chosen θ_s. This result, which appears here as generalization of (5.12), is actually a consequence of the remarkable generalization of (5.7), which Crocco in (16) has also shown to be approximately true. His calculations indicate that *t is a linear function of t_w as well as of M^2 the coefficients of which depend on u (and on σ), but only very weakly on θ_s.* Crocco's discovery partly justifies the fact that boundary layer investigations are usually not undertaken on the basis of the correct Sutherland law (which would introduce the absolute temperature of the standard state, say T_∞, as an additional physical parameter): the temperature field function $T(u)/T_\infty$ depends on the boundary conditions T_w and T_∞ principally through T_w/T_∞, when $\sigma \sim 1$. (This is by no means true for the stress field $K(u)$.)

Regarding the dependence of the temperature field on σ, we have already noted Pohlhausen's result for vanishing heat transfer in the case of constant $\bar{\rho}$ and $\bar{\mu}$. Then equation (4.117) holds, which we may write

$$t_w - 1 \sim \tfrac{1}{2}(\gamma - 1)M^2\sqrt{\sigma} \tag{5.15}$$

but this case is obviously covered by Crocco's case $\omega = 1$, although his theory is concerned with the variability of the product $\bar{\rho}\bar{\mu}$ rather than with the individual variabilities of $\bar{\rho}$ and $\bar{\mu}$. In other words, Pohlhausen's case is identical with Crocco's case $\omega = 1$, $(dt/du)_w = 0$, equation (5.12) holds strictly and implies together with (5.15)

$$J(1) = \frac{t_w - 1}{\sigma(\gamma - 1)M^2} \sim \frac{1}{2\sqrt{\sigma}} \quad \text{by (5.15)} \tag{5.16}$$

On replacing $J(1)$ in (5.12) by $1/(2\sqrt{\sigma})$, it is seen that the approximation (5.15) is generally good whenever $(dt/du)_w = 0$ and $\omega = 1$. Now from Emmons and Brainerd's work (15) we know that (5.15) holds also for $\omega = 0.768$. To interpret this result, let us write equation (5.7) for the particular case of $(t_u)_w = 0$. Using (5.12) and (5.10) we obtain

$$t(u) = t_w - \sigma J(u,\sigma)(\gamma - 1)M^2 \tag{5.17}$$

valid for $\omega = 1$, where $J(u,\sigma)$ is $O(u^2)$ for $u \sim 0$, as stated before; equation (5.16), on the other hand, should be obtained from (5.17) by putting $u = 1$, *but holds also for $\omega \neq 1$.* This makes it practically certain that

an approximately correct form of (5.17) for any ω is achieved when one puts

$$J(u,\sigma) = \frac{u^2}{2\sqrt{\sigma}} \tag{5.18}$$

In other words, for vanishing heat transfer,

$$t + \tfrac{1}{2}\sqrt{\sigma}\,(\gamma - 1)M^2u^2 = \text{constant} \tag{5.19}$$

is an approximate integral of the energy equation in the restricted problem, whatever the viscosity exponent. [Cf. (4.117a) and the subsequent remark.]

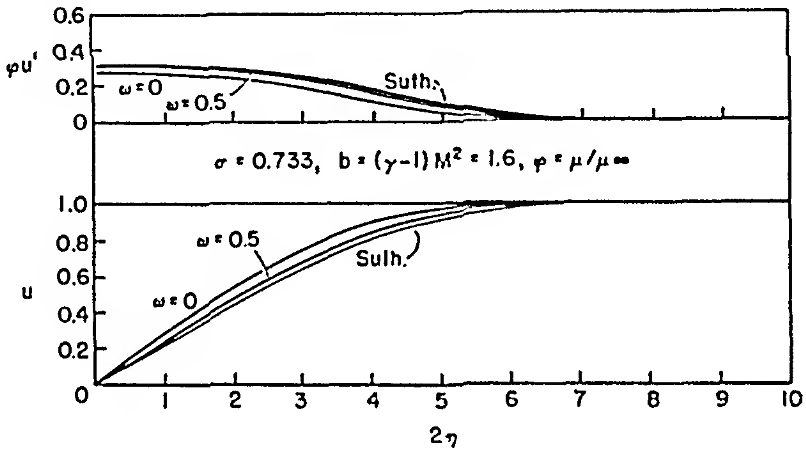


FIG. 12a. Variation of u and $\varphi u'$ for various φ's when $\sigma = 0.733$ and $b = 1.6$.

It is interesting to look at some of the results of (15) from this point of view. Before choosing $\omega = 0.768$ for their calculations, Emmons and Brainerd investigate, for fixed σ and M, the effect of different viscosity laws on the u- and t-profiles. The curves obtained for $\omega = 0$ and 0.5, and according to (4.12a) for $T_\infty = 650°$K are reproduced in Fig. 12a,b. For both profiles the effect of the changed viscosity law is quite noticeable, but the curves become practically identical when η is eliminated by plotting $(t - 1)/(b/2)$ vs u.

From the same paper we also reproduce Fig. 13, a set of profiles of the velocity component normal to the plate. Most papers present u-profiles only, hence it appears worth while to show for once the general behavior of the other component. In comparing Fig. 13 with Fig. 12a it must be considered that $v/u = (V/U)\ \mathrm{Re}^{1/2}\ (x/L)^{1/2}$: the scale of v is strongly exaggerated except in the vicinity of the leading edge.

Returning to our discussion of the temperature field, the main result must be seen in the extension of the validity of formula (5.7) to any ω

(or θ_s) value. This implies the extended validity of Reynolds' analogy (5.11), which is a matter of considerable practical interest, especially if one can find a simple approximation for the dependence of the coefficient $I(1)$ on σ. [We already know $J(1) \sim 1/(2\sqrt{\sigma})$.]

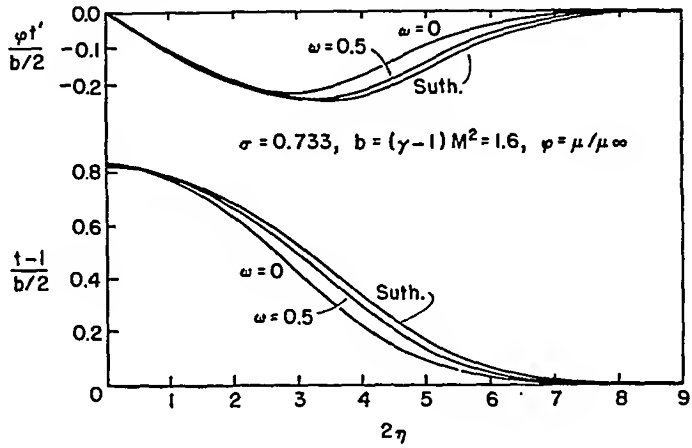


FIG. 12b. Variation of t and $\varphi t'$ for various φ's when $\sigma = 0.733$ and $b = 1.6$.

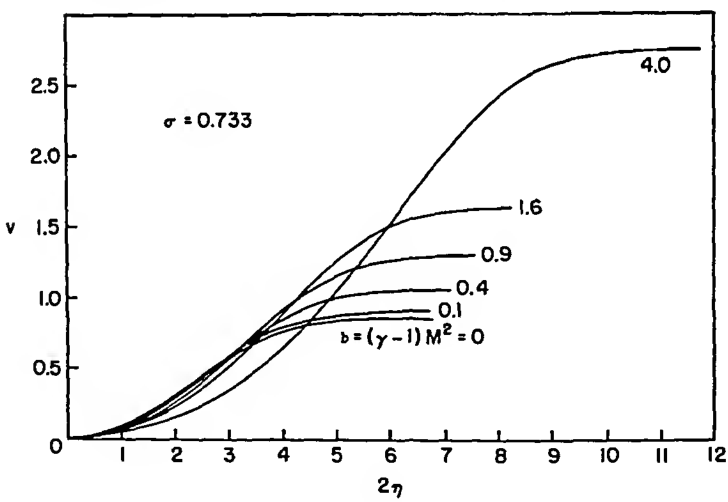


FIG. 13. Variation of v in boundary layer.

Here we can again argue as on p. 78 that (5.11) holds for $\omega = 1$, and identify this case with Pohlhausen's calculation of the heat transfer to a plate at prescribed temperature T_w, in a fluid with constant ρ and μ *when the heat produced by friction in the fluid is neglected.* This means solving (4.114b) with $u = \zeta'/2$ under omission of the inhomogeneous term, hence $M = 0$. We then obtain from (5.11)

$$Q_w = F_w \left(\frac{T_\infty}{U_\infty}\right)\left(\frac{k}{\mu}\right)\frac{(t_w - 1)}{I(1)} \tag{5.20}$$

and from ref. (5)

$$Q_w = f(\sigma)k\sqrt{\mathrm{Re}}\, T_\infty(t_w - 1)$$

with

$$1/f(\sigma) = \int_0^\infty \exp\left(-\sigma \int_0^\eta \zeta\, d\eta\right) d\eta \tag{5.20'}$$

But $F_w = (\alpha/\sqrt{\mathrm{Re}})(\frac{1}{2}\rho U_\infty^2)$ according to (3.20), hence, by comparison of (5.20) and (5.20′)

$$\frac{1}{I(1)} = \frac{2f(\sigma)}{\alpha}, \quad (\alpha = 1.328) \tag{5.21}$$

Apparently on the basis of the numerical evaluation of $f(\sigma)$ Pohlhausen states that

$$f(\sigma) \sim (\alpha/2)\sigma^{1/3}, \quad 0.6 < \sigma < 15 \tag{5.22}$$

In (1), p. 627, this approximation is shown to evolve, when instead of $\frac{1}{2}\zeta'$ a quartic approximation of u is used in solving (4.114b). Another way to (5.22) starts from the relation

$$\int_0^\infty (\zeta'')^\sigma d\eta \sim 2\alpha^{\sigma-1}\sigma^{-1/3} \tag{5.23}$$

which is correct within 1% for $0.6 < \sigma < 1$. The latter formula follows readily from the representation of ζ'' in the neighborhood of zero given in (4.123), which reads $\zeta'' \sim \alpha \exp(-\alpha\eta^3/6)$ in the Blasius variable η. Then (5.22) follows on replacing in the second equation (5.20′) $-\int_0^\eta \zeta\, d\eta$ by $\log(\zeta''/\alpha)$ (see p. 66) and applying (5.23).

Equations (5.21) and (5.22) yield the desired approximation

$$\frac{1}{I(1)} \sim \sigma^{1/3} \tag{5.24'}$$

hence the other coefficient in (5.11) becomes

$$\frac{\sigma J(1)}{I(1)} \sim \frac{1}{2}\sigma^{5/6}. \tag{5.24''}$$

We now *define* the equilibrium temperature T_e of a gas flow in heat exchange with the wall as T_w in the case of no heat exchange. Then, by (5.15)

$$t_e - 1 \sim \tfrac{1}{2}(\gamma - 1)M^2\sqrt{\sigma} \tag{5.25}$$

Using the approximations for the coefficients (5.24) together with (5.25), Reynolds' analogy (5.11) may be put in the following very compre-

where $b = (\gamma - 1)M^2$ as before, and works in the variable σ by adding the factor $\sqrt{\sigma}$ to b. In this way the formula

$$c_{\tau_0} \sqrt{\text{Re}} \sqrt{x/L} \sim 0.664[1 + 0.3b \sqrt{\sigma}]^{(1-\omega)/2} \tag{5.28}$$

is obtained, valid for $(T_y)_w = 0$.

In the general case, Young's starting point is an elliptic approximation of $K(u)$,

$$K(u) = K_0 \sqrt{1 - u^2}$$

Substituting $\bar{\mu}\bar{\rho} = [t(u)]^{\omega-1}$ in Crocco's equation (4.31a) and writing this equation as an integral equation for $K(u)$, one is led to the relation

$$K_0^2 \sim \int_0^1 du_1 \int_0^{u_1} \frac{t^{\omega-1} 2u\,du}{\sqrt{1 - u^2}} \tag{5.29}$$

which is evaluated for $\sigma = 1$, in which case $t(u) =$ quadric in u. This step, which includes expansion of the term $t^{\omega-1}$, provides the necessary information about the form of K_0 as function of the parameters ω, and b and t_w which are hidden in t. The σ-dependence is again worked in through replacing b by $b\sqrt{\sigma}$. Finally, the constants obtained in the integration (5.29) are slightly adjusted for best fit of the known results. Young obtains

$$c_{\tau_0} \sqrt{\text{Re}} \sqrt{x/L} \sim 0.664(0.45 + 0.55t_w + 0.09b \sqrt{\sigma})^{(\omega-1)/2} \tag{5.30$'$}$$

where it is essential that the first two coefficients inside the parenthesis add up to 1. The formula represents the known numerical results of (11,14,15,16,18) very well in the case $(T_y)_w = 0$; it becomes identical with (5.28), except for the coefficient of $b\sqrt{\sigma}$, which is now 0.365. For prescribed T_w, (5.30′) represents Crocco's results within 1%.

Another comprehensive representation of the wall stress function in the restricted problem has been recently proposed by Johnson and Rubesin (27). It is based on a survey of the numerical results of (11,13,16) and lumps the effect of wall temperature and Mach number in a formula not explicitly depending on σ which reads in the present notation

$$c_{\tau_0} \sqrt{\text{Re}} \sqrt{x/L} \sim 0.664(0.42 + 0.58t_w + 0.079b)^{(\omega-1)/2} \tag{5.30$''$}$$

Relations (5.27) and (5.30) answer the questions of heat and momentum transfer in the restricted problem. A further concern of practical importance is the relation between M and t_w that must hold to insure *actual* cooling of the gas stream by a wall. Young compares in (25) the local dissipation in the boundary layer with the local heat transfer on the

basis of the preceding results and arrives at an approximate formula for the Mach number M_1 at which transition from cooling to heating occurs:

$$M_1{}^2 = \frac{2}{\gamma - 1} \frac{1 - t_w}{\sigma^{2/3}\left(\frac{\pi}{2} - \sigma^{-1/6}\right)} \tag{5.31}$$

The same problem had been solved previously in the case $t_w = 0.25$ by Hantzsche and Wendt for $\omega = 1$ and by Kármán and Tsien for

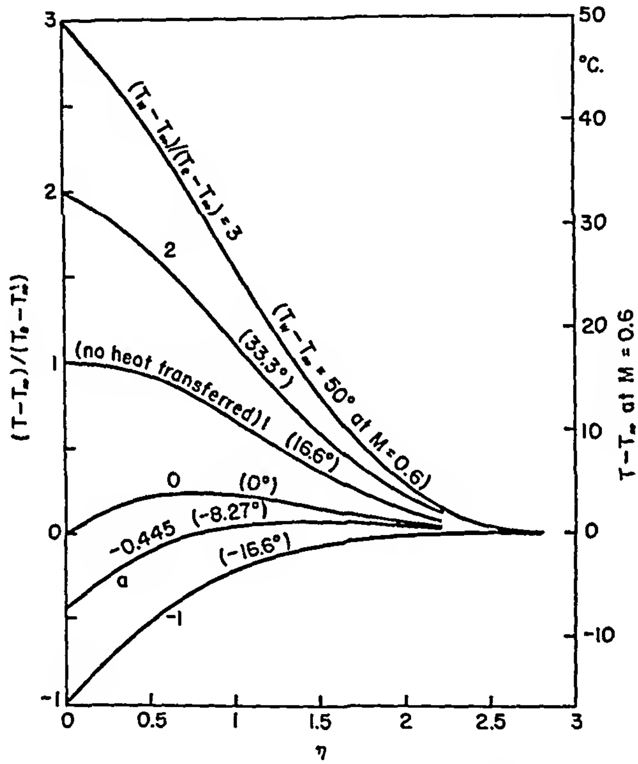


FIG. 14. Temperature field for $\sigma = 0.7$ disregarding the variability of μ and ρ (equivalent to restricted problem with $\omega = 1$).

$\omega = 0.76$ (cf. p. 34), in both cases $\sigma = 1$ is assumed. Their values for M_1 are in substantial agreement with (5.31). For a further discussion of the results of these authors see also the boundary layer report by Carrier and Lewis (28).

We conclude this section with a brief reference to the approximate treatments of the restricted problem by Eckert and Drewitz (22) and Schuh (29). The former apply Pohlhausen's method of (5) to the case of prescribed T_w *without disregarding the heat produced within the boundary layer*. (This is the complete equation (4.114b) with $u = \frac{1}{2}\zeta'$ and pre-

scribed T_w.) The results, although superseded by Crocco's computations, are of practical interest because the temperature profiles are represented directly in the independent variable η and referred to the excess of the equilibrium temperature $T_e - T_\infty$ or $T_\infty \sqrt{\sigma}\, b/2$ by (5.25′). Figure 14 shows the temperature field obtained for $\sigma = 0.7$. The curves $(t - 1)/(t_e - 1)$ vs η are plotted for a set of boundary conditions $(t_w - 1)/(t_e - 1) = -1, 0, 1, 2, 3$. The conditions for the curve a are such that the entire heat produced by friction within the boundary layer is carried away through the wall. Results are accurate within a few percent up to $M = 2$, and for moderate temperature differences. For $M = 0.6$, the diagram can be read directly in temperature differences, when the scale to the right is used together with the labels set in parenthesis.

Schuh's work belongs here principally for methodical reasons. In his paper (29) the integration method of Piercy and Preston[34] is employed. Starting from the boundary layer equations in the form (4.112)–(4.114) (which we now write with the independent *Blasius variable* η) we can rewrite that system in terms of the function $f(\eta) = \xi/\theta$:

$$\begin{aligned} &\frac{d}{d\eta}(\varphi u') + (\varphi u')f\varphi^{-1} = 0 \qquad \cdots [4.113] \\ &\frac{d}{d\eta}(\varphi\theta') + (\varphi\theta')\sigma f\varphi^{-1} = -\sigma b\varphi u'^2 \cdots [4.114'] \end{aligned} \tag{5.32}$$

where $f(\eta)$ can be written in terms of the "physical" variables

$$f(\eta) = 2\int_0^\eta \left(\frac{\rho}{\rho_\infty}\right) u\, d\eta \quad \cdots [4.112] \tag{5.32'}$$

In order to have the same type of boundary conditions for u and the temperature function, we introduce

$$\bar{\theta} = \frac{T - T_w}{T_\infty - T_w} = \frac{\theta}{1 - \theta_w} + \text{constant}$$

in the second equation (5.32); this adds to the right member the factor $1/(1 - \theta_w)$. Then the boundary conditions are

$$u(0) = \bar{\theta}(0) = 0, \qquad u(\infty) = \bar{\theta}(\infty) = 1$$

The solution of (5.32) can now be written in terms of $f(\eta)$, using the abbreviation $P(\eta) = \int_0^\eta f\varphi^{-1}d\eta$, viz.,

[34] *Phil. Mag.* [7], 21 (1936).

$$u = \frac{J(\eta)}{J(\infty)} \tag{5.33a}$$

$$\tilde{\theta} = \frac{1 + B(\infty)}{A(\infty)} A(\eta) - B(\eta) \tag{5.33b}$$

where

$$J(\eta) = \int_0^\eta \varphi^{-1} \exp(-P) d\eta, \qquad A(\eta) = \int_0^\eta \varphi^{-1} \exp(-\sigma P) d\eta$$

and

$$B = \frac{\sigma(\gamma - 1)M^2}{1 - \theta_w} \int_0^\eta \varphi^{-1} \exp(-\sigma P) \left[\int_0^\eta \varphi u'^2 \exp(\sigma P) d\eta \right] d\eta$$

On the basis of (5.32′) and (5.33) the following iteration process can be devised. Using the known solutions for constant ρ and μ, that is, $\frac{1}{2}\zeta'$ for u_0, and Eckert and Drewitz's solution for θ_0 with the proper θ_w, one computes $f_0(\eta)$ and $P_0(\eta)$, which determines the improved u_1 according to (5.33a) and θ_1 according to (5.33b). The next step starts in the same way from u_1 and θ_1.

In (29) this process is applied to an airstream along a hot plate, but under omission of the dissipation term in the second equation (5.32) (hence $B \equiv 0$), for $\theta_w = 1.25$, 1.5, and 3.39, assuming $\omega = 0.78$ and a *slightly different* exponent for k so that σ varies slightly too. The same method is applied to a diffusion problem. This is possible because the energy equation without dissipation term is formally equivalent to the equation of diffusion.

4. Other Problems

In this section Chapman and Rubesin's treatment of the case $T_w = T_w(x)$, $p_x = 0$ (19), and Tifford's analysis of the case $U_\infty \propto x^m$ is briefly reviewed. A remark on wakes and jets and another on the axisymmetric boundary layer follow.

The point of departure in (19) is the linear equation (4.17), in which ζ is a known function of η; it can be solved by well known methods. Again, it is an advantage to exhibit the thermometer temperature $t_e = T_e/T_\infty$ in the boundary conditions:

$$t(\xi,0) = t_e + t(\xi), \qquad t(\xi,\infty) = 1 \tag{5.34}$$

A particular solution of (4.17) which depends only on η can be easily found by quadratures. When in (4.17) $\partial t/\partial \xi$ is assumed to vanish, there simply remains (4.114b) with $u = \frac{1}{2}\zeta'$. Let now $N(\eta)$ be that solution for which $N'(0) = 0$ and $N(\infty) = 1$. Then

$$N(\eta) = 1 + (b/2)r(\eta), \qquad r(\eta) = \tfrac{1}{2}\sigma \int_\eta^\infty (\zeta'')^\sigma d\eta_1 \int^{\eta_1} (\zeta'')^{2-\sigma} d\eta_2 \tag{5.35}$$

cf. (4.117″), and $N_w = t_e$, therefore $r(0)$ is the recovery factor in agreement with (5.25′).

The solution of the homogeneous equation associated with (4.17) is sought in the separated form[35]

$$t = X(\xi)Y(\eta) \tag{5.36}$$

The functions $X_n(\xi)$ are simply powers of ξ, while the Y_n satisfy the equation

$$Y_n'' + \sigma\zeta Y_n' - 2n\sigma\zeta' Y_n = 0 \tag{5.37}$$

with the boundary conditions $Y_n(0) = 1$, $Y_n(\infty) = 0$. This equation is solved by numerical methods (using the known asymptotic behavior of ζ, the asymptotic solution of (5.37) can be written up in terms of Hermite polynomials). Graphs of the first five functions Y_n ($n = 0$, $1 \cdots 4$) are supplied in (19).

The complete solution of equation (4.17) is

$$t(\xi,\eta) = N(\eta) + \sum_0 a_n \xi^n Y_n(\eta) \tag{5.38}$$

and the a_n are found from the series expansion of the given distribution $t(\xi)$:

$$t(\xi) = \sum_0 a_n \xi^n \tag{5.39}$$

One returns to the physical variable y according to (4.18a) in which the solution (5.38) must be substituted; thus y depends on the integrals of $r(\eta)$ and $Y_n(\eta)$, graphs of which are supplied in (19), where also the formula for q_w is given and the local heat transfer coefficient $\beta^*_{\text{loc}} = q_w(T_w - T_e)$ is computed. Using the approximation (5.23) for $\int_0^\infty (\zeta'')^\sigma d\eta$, the Nusselt number could be written in the form

$$\frac{\text{Nu}^*}{\sqrt{\text{Re}}} = \frac{C_{D_f}}{2}\sqrt{\text{Re}}\,\sigma^{1/3} + \frac{\sqrt{C}}{2(t_w - t_e)}F \tag{5.40}$$

where F is a function of ξ, which depends on the a_n of (5.39) and the $Y_n'(0)$, and C is the coefficient introduced in (4.13), to be taken for an average value of T_w. In the first term, (5.40) coincides, of course, with (5.27). An example is discussed in detail in which $t(\xi) = t_w - t_e$ is assumed as $t_e \times$ quadric (ξ), while $C = 1$ for simplicity. This makes a_n proportional to t_e and thus dependent on M, and therefore

[35] X, Y, N is the original notation which has been kept here, although these quantities are, of course, dimensionless.

facilitates the discussion of the influence of M (appearing directly in t_e by $N(0)$, and in the return from η to y); M is taken as 0.5 and 3; the numbers a_n/t_e are chosen such that $T_w/T_e = 1 +$ quadric (ξ) drops from 1.25 to 0.75 over the length of the plate.

The approach of (5 and 22) has been generalized by Tifford in (26) for *nonvanishing pressure gradient*, although Pohlhausen's method seems perhaps less justified in this case. For uniform free stream the boundary layer equations depend only on the product $\rho\mu$, but the additional term p_x involves a factor $1/\rho$ (e.g., p. 46). The conditions of operation for (26) must be formulated as follows: ρ and μ are individually constant; and the variability of ρ along the boundary layer edge is disregarded in spite of $p_x \neq 0$ *and* $T_x \neq 0$. When this is accepted and a free stream velocity $U_\infty = cx^m$ is assumed ($m > 0$, flow downstream of the stagnation point), then the accurately known solution of the Falkner-Skan equation,[36] $F(\bar{\eta})$, can be employed in the same way as Blasius solution ζ in the case $p_x = 0$. But the complication is that the energy equation remains a partial differential equation (of course linear and of second order, and inhomogeneous) even after introduction of the (modified) Blasius variable $\bar{\eta} = [\frac{1}{2}(m + 1)U_\infty/x\nu]^{1/2}y$, whereas for $p_x = 0$ the ordinary equation (1.114b) with $u = \frac{1}{2}\zeta'$ is obtained. The other independent variable in Tifford's analysis is x, or rather a function of it, viz., $\xi = (T_\infty - T_w)/(U_\infty{}^2/c_p)$, where T_∞ and U_∞ are known functions of x, and T_w a given constant. The unknown function $\theta = (T - T_w)/(T_\infty - T_w)$.

It is possible to split θ in two parts

$$\theta = \Phi(\bar{\eta},\xi) + \Psi(\bar{\eta})/\xi \tag{5.41}$$

so that Φ satisfies a *homogeneous* equation that can be separated into two ordinary equations. One obtains in this way

$$\Phi = \frac{X(\xi)Y(\bar{\eta})}{Y(\infty)} \tag{5.42}$$

where

$$X(\xi) = 2(T_\infty{}^* - T_w)/(T_\infty - T_w) \text{ and } Y(\bar{\eta}) = \int_0^{\bar{\eta}} d\bar{\eta}_1, \exp\left(-\sigma \int_0^{\bar{\eta}_1} F d\bar{\eta}_2\right).$$

The function $\Psi(\bar{\eta}) + \frac{1}{2}$ must be determined from an inhomogeneous second order equation whose coefficients are σF, $-2\beta\sigma F'$, and $-\sigma(F'')^2$ on the right hand, by numerical methods, β stands for $2m/(m + 1)$.

The physical significance of the decomposition (5.41) can be seen as follows. In the analogous case of equation (4.114b) the term $\sigma(\gamma -$

[36] Hartree, *Proc. Cambridge Phil. Soc.*, **33**, 223 (1937). The equation reads $F''' + FF'' = \beta(F'^2 - 1)$, with $F(0) = F'(0) = 0$, $F'(\infty) = 1$.

$1)M^2u'^2$ that makes that equation inhomogeneous stems from the term μU_y^2 in equation (2.20), the only dissipative term left in the boundary layer energy equation. Thus it is correct to say that the *homogeneous* equation associated with (4.114b) determines the temperature field under disregard of the heat created by the work of the friction forces; and in a strictly incompressible theory their work is the only source of heat within the fluid. In the present case the interpretation follows by analogy: Φ is the temperature field under disregard of the heat internally produced.

The effect of σ on the local heat transfer coefficient based on $T_\infty^* - T_w$ is shown numerically in the cases $m = 0, 1$, omitting the Ψ-effect. The latter may, in general, be taken into account as a correction term that adds to the given T_∞^* and depends on σ and m.

The theory is now generalized for arbitrary body shape by series expansion of U_∞. The quest for a relation between dynamic and thermodynamic parameters leads then to an approximate formula that correlates q_w with the gradient of the momentum loss $d(\rho U_\infty^2\theta)/dx$ rather than with τ_w, θ being the momentum thickness defined in (4.74) for constant ρ. Values of $\beta_{1\infty}^{**} = q_w/(T_\infty^* - T_w)$ following from this relation are compared with those determined from the Φ-distribution in the case of a wedge with varying angle $\beta\pi$ for $m = 0, \frac{1}{5}, \frac{1}{3}, 1$, at $\sigma = 0.7$, and the approximate formula is applied to compute the heat transfer for the front half of a circular cylinder up to $\pm 70°$.

Flow in wakes and jets is treated in (17). The standard incompressible solution $w = w_0 \exp\ (-\frac{1}{2}\eta^2)$ is again used $(w = 1 - u)$. Howarth finds that the incompressible theory underestimates the wake thickness by $\frac{1}{2}bM^2D/\rho_\infty U_\infty^2$ where D is the drag force per unit of breadth of the (symmetrical) obstacle. In the same way the actual width of a jet is found smaller by the amount $[(\gamma - 1)/2a_0^2]F/\rho_0$ where F is the flux of momentum through the jet (subscript 0 refers to the gas at rest). Another aspect of the jet problem is treated in (21); again $\sigma = 1$, but Chapman treats both cases, $\omega = 0.76$ and $\mu = Ct$. The method is that of (11) and is presented in great detail; velocity profiles for the mixing layer are shown for $M = 0, \cdots, 5$ and, in the case of the linear viscosity law, for $T_\infty = 400°$R.

The theorem of the mathematical equivalence of axisymmetric and plane boundary layer flow is due to Mangler who published it first in 1945. It is reviewed in (28), Appendix II. The transformation becomes possible, because the equations of motion and energy of the axisymmetric boundary layer are formally identical with our equation (2.17) and (2.20), provided one replaces x, y by s, n and U, V by q_s, q_n, where s is the arc length along the meridian curve of the surface of revolution and n

the distance along the normal. But the continuity equation reads

$$\frac{\partial}{\partial s}(\rho r_0 q_s) + \frac{\partial}{\partial n}(\rho r_0 v_n) = 0 \tag{5.43}$$

where $r_0(s)$ is the distance of a surface point from the axis of revolution. Complete formal identity can be established, when a point s, n in the axisymmetric boundary layer is associated with a point x, y in a plane boundary layer by

$$x = \int_0^s \left(\frac{r_0}{L}\right)^2 ds, \quad y = \left(\frac{r_0}{L}\right) n \tag{5.44}$$

where L is a fixed reference length. The values of p, T, ρ, μ are the same in associated points.

In the case of a cone in supersonic flow (but only with *attached* shock wave), the associated plane flow becomes simply our restricted problem, since p_∞ is constant. In a later publication (30) Mangler discusses some examples of *incompressible* axisymmetric boundary layer flows, among them again the cone; but here $q_\infty(s) \propto s^m$, and the associated plane problem is determined by the Falkner-Skan equation (footnote 36) with $m/3$.

Note added in proof. Several contributions to our subject appeared while this paper was in press, among them are:

Stalder, J. R., Rubesin, M. W., and Tendeland, T., A determination of the laminar-, transitional-, and turbulent-boundary-layer temperature-recovery factors on a flat plate in supersonic flow, N.A.C.A. T.N. 2077. This is the first precision determination of the recovery factor r for air, undertaken with the aim of checking the $\sqrt{\sigma}$-theory. In the laminar range of the experiments, $0.2 \times 10^6 < \text{Re} < 1.3 \times 10^6$, with $M = 2.4$ and $T_\infty{}^* = 560°\text{R}$ (100°F), the result was $r = 0.881$, that is 1.5% higher than the free stream value of $\sqrt{\sigma}$.

Illingworth, C. R., Steady flow in the laminar boundary layer of a gas, Proc. Roy. Soc. (London), A199, 533 (1949). In this paper the influence of gravity on boundary layer flow is discussed, when the generators of the cylindrical surface of the obstacle, which are parallel to the z — axis, are perpendicular to the incident stream, but not necessarily horizontal. From a discussion of the corresponding three-dimensional boundary layer equations a condition is developed for the primary flow (for which $\partial/\partial z = 0$, $g = 0$) to be a reasonable approximation to the actual flow. This is applied to horizontal and vertical cylindrical obstacles; in the latter case a method of computing the cross flow due to gravity is given.

In his discussion of the primary flow in a cylindrical boundary layer, Illingworth uses the Mises transformation (Eqs. 4.6, 7), modified as indicated in the first two paragraphs of IV, IC (Howarth). When Howarth's assumptions $\sigma = \omega = 1$, $(T_v)_w = 0$ are adopted, and the x-coordinate is stretched by the introduction of the independent variable $s = \int_0^x \mu\rho u_\infty dx$, it becomes possible to associate to a compressible boundary layer flow with given u_∞^c-law an incompressible boundary layer flow with a

certain u_∞^{inc}-law such that $u^c = u^{\text{inc}}$ at points with same s and ψ values. The mapping of the boundaries is given by the two equations $x^{\text{inc}} = \int_0^{x^c} [1 + \frac{1}{2}b(x)]^n dx$, $u_\infty^{\text{inc}}(x^{\text{inc}}) = a_{\infty,s} M_\infty(x)$, where $n = -\frac{1}{2}(3\gamma - 1)/(\gamma - 1)$.—The same result was obtained by S. Christensen in his Harvard thesis, 1951, and by K. Stewartson in Correlated incompressible and compressible boundary layers, Proc. Roy. Soc. (London), A200, 84 (1949), where it is applied (among other things) to estimate separation in flow with linear velocity lapse (cf. p. 56). Illingworth paper has also a section on the restricted problem with $\omega \neq 1$, for which an extension of the method of (11) is proposed. This, however, coincides in its essentials with Schuh's method in (29).

Several of the papers referred to have recently been made more easily accessible, viz.

(9) is translated as N.A.C.A. Tech. Memo. No. 1256.

(29) is translated as N.A.C.A. Tech. Memo. No. 1275.

(25) has appeared in a more complete version in the Aeronautical Quarterly (RAS London) **1**, 137 (1949).

References

1. Modern developments in fluid dynamics, ed. S. Goldstein, Oxford, Clarendon Press, 1938.
2. Mises, R. von, Bemerkungen zur Hydrodynamik, *Z. angew. Math. Mech.*, **7**, 425 (1927).
3. Die Differential- und Integralgleichungen der Mechanik und Physik, ed. Ph. Frank and R. von Mises, Braunschweig, 1935, Vol. 2, p. 515 ff.
4. Tetervin, N., Remarks concerning the behavior of the laminar boundary layer in compressible flows, *N.A.C.A. Tech. Note*, No. 1805, 1949.
5. Pohlhausen, E., Der Wärmeaustausch zwischen festen Körpern und Flüssigkeiten mit kleiner Reibung und kleiner Wärmeleitung, *Z. angew. Math. Mech.*, **1**, 121 (1921).
6. Handbuch der Experimentalphysik, ed. W. Wien and F. Harms, Leipzig, 1931, Vol. IV, 1, p. 365 f.
7. Crocco, L., Sulla transmissione del calore da una lamina piana a un fluido scorrente ad alta velocità, L'Aerotecnica, Vol. XII (1932); translated as *N.A.C.A. Tech. Memo.*, No. 690, 1932.
8. Busemann, A., Gasströmung mit laminarer Grenzschicht entlang einer Platte, *Z. angew. Math. Mech.*, **15**, 23 (1935).
9. Blasius, H., Grenzschichten in Flüssigkeiten mit kleiner Reibung, Göttingen Thesis 1907, also *Z. Math. Phys.*, **56**, 1 (1908).
10. *Atti dei congressi Volta*, **5**, 1935 (5th Volta Congress), p. 226 ff (English), Reale Academia, Roma.
11. Kármán, Th. von, and Tsien, H. S., Boundary layer in compressible fluids, *J. Aeronaut Sci.*, **5**, 227 (1938).
12. Crocco, L., Sullo strato limite laminare nei gas lungo una parete piana, *Rend. Circ. Matem. Palermo*, **63**, 121 (1941), also Una caratteristica trasformazione delle equazioni dello strato limite nei gas, Atto di Guidonia, No. 7, 1939.
13. Hantzsche, W., and Wendt, H., Zum Kompressibilitätseinfluss bei der laminaren Grenzschicht der ebenen Platte, in Jahrbuch d. deutschen Luftfahrtforschung, 1940, I, p. 517, abstracted in Report No. F-TS-2072-Re (1948), Air Mat. Com., Dayton, Ohio.

14. Emmons, H. W., and Brainerd, G. J., Temperature effects in a laminar compressible-fluid boundary layer along a flat plate, *J. Applied Mechanics*, **8**, A-105 (1941).
15. Brainerd, G. J., and Emmons, H. W., Effect of variable viscosity on boundary layers, with a discussion of drag measurements, *J. Applied Mechanics* **9**, A-1 (1942).
16. Crocco, L., Lo strato limite laminare nei gas, Monografie scientifiche di aeronautica, No. 3, 1946, Assoc. Cult. Aeron., Roma, translated by I. Hodes *et al.*, Aerophysics Laboratory, as Report CF-1038, North American Aviation Inc., and also as R.A.E., Library Translation No. 218.
17. Howarth, L., Concerning the effect of compressibility on laminar boundary layers and their separation, *Proc. Roy. Soc.* (*London*), **A194**, 16 (1948).
18. Cope, W. F., and Hartree, D. R., The laminar boundary layer in compressible flow, *Phil. Trans. Roy. Soc.* London, **A241**, No. 827, 1 (1948).
19. Chapman, D. R., and Rubesin, M. W., Temperature and velocity profiles in the compressible laminar boundary layer with arbitrary distribution of surface temperature, *J. Aeronaut. Sci.*, **16**, 547 (1949).
20. Kalikhman, B. L., Heat transmission in the boundary layer, *N.A.C.A. Tech. Memo.*, **No. 1229**, 1949, translation of "Gazodinamicheckaya Teoriya Teploperedachi" in Prikladnaya Matematika i Mekhanika, **10**, 1946.
21. Chapman, D. R., Laminar mixing of a compressible fluid, *N.A.C.A. Tech. Note*, **No. 1800**, 1948.
22. Eckert, E., and Drewitz, O., Der Wärmeübergang an eine mit grosser Geschwindigkeit längs angeströmte Platte, *Forsch. a.d.G.d. Ingenieurw.* **A11**, 116 (1940); transl. as *N.A.C.A. Tech. Memo.*, **No. 1045**, 1943.
23. Meksyn, D., The laminar boundary layer equations, II. Integration of nonlinear ordinary differential equations, *Proc. Roy. Soc.* (*London*), **A192**, 568 (1948).
24. —— Integration of the boundary layer equations for a plane in a compressible fluid, *Proc. Roy. Soc.* (*London*), **A195**, 180 (1948).
25. Young, A. D., Skin friction in the laminar boundary layer in compressible flow, Rept. No. 20, The College of Aeronautics, Cranfield, 1948.
26. Tifford, A. N., The thermodynamics of the laminar boundary layer of a heated body in a high-speed gas flow field, *J. Aeronaut. Sci.*, **12**, 241 (1945).
27. Johnson, H. A., and Rubesin, M. W., Aerodynamic heating and convective heat transfer-summary of Literature survey, *Trans. Am. Soc. Mech. Engrs.*, **71**, 447 (1949).
28. Lewis, J. A. (G. F. Carrier, superv.), Boundary layer in compressible fluid, Tech. Rept. No. F-TR-1179-ND (1948), Air Mat. Com., Wright Field, Ohio.
29. Schuh, H., Einige Probleme der Strömung des Wärmeüberganges und der Diffusion bei Laminarströmung längs einer ebenen Platte, *Österr. Ing.-Archiv*, **2**, 346 (1948), also *Z. angew. Math. Mech.*, **25/27** (1947).
30. Mangler, W., Zusammenhang zwischen ebenen und rotationssymmetrischen Grenzschichten in kompressiblen Flüssigkeiten, *Z. angew. Math. Mech.*, **28**, 97 (1948).

Bending of Curved Tubes*

By R. A. Clark and E. Reissner

Massachusetts Institute of Technology, Cambridge, Massachusetts

CONTENTS

I. Introduction 93
II. Formulation of the Problem 97
III. The Equations for Bending of a Tube with Uniform Circular Cross Section 99
IV. Trigonometric Series Solution for the Tube with Uniform Circular Cross Section 102
1. No β-terms, One ψ-Term Retained 104
2. One β-Term, One ψ-Term Retained 105
3. One β-Term, Two ψ-Terms Retained 105
4. Two β-Terms, Two ψ-Terms Retained 107
V. Asymptotic Solution for the Tube with Uniform Circular Cross Section 109
VI. The Equations for Bending of a Tube with Uniform Elliptical Cross Section 117
VII. Asymptotic Solution for the Tube with Uniform Elliptical Cross Section 118
References 122

I. Introduction

In 1910 A. Bantlin (1) found experimentally that a curved tube is much more flexible in bending than a straight tube of the same cross section. The following year von Kármán (6) gave a theoretical explanation of this phenomenon. Briefly, for a curved tube there is a tendency for the cross section to flatten because of the continual change of direction of the stresses which are parallel to the center line of the tube and which balance the applied moment. When flattening takes place, the strain in the outermost fibers of the tube, for a given change of curvature of the center line of the tube, is less than it would be if there were no flattening of the cross sections. Consequently a smaller bending moment is required to produce a given change of curvature. Defining rigidity of the tube as the ratio of bending moment to change of curvature it follows that flattening of the cross sections reduces the rigidity of the tube.

The situation may be illustrated by means of Fig. 1 which shows a tube in its undeformed and its deformed state. Let a be the radius of the center line of the undeformed tube, $a\theta$ the length of the center line, l

* Written with the support of the Office of Naval Research under Contract N5 ori-07834.

the length of the outermost fiber, and b the distance of the outermost fiber from the center line. Let $a - \Delta a$, $\theta + \Delta\theta$, $l + \Delta l$, and $b - \Delta b$ be the corresponding quantities for the deformed tube. Neglecting products of small quantities, let

$$\Delta K = \frac{1}{a - \Delta a} - \frac{1}{a} = \frac{\Delta a}{a^2} \tag{1.1}$$

be the change of curvature of the center line of the tube. Since the center line remains unchanged in length we have the relation $(a - \Delta a)$

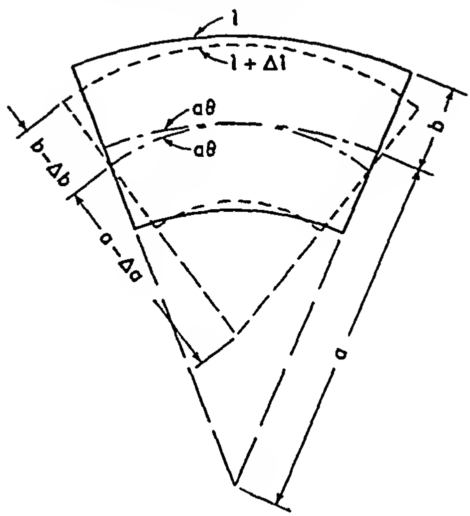


FIG. 1

$(\theta + \Delta\theta) = a\theta$ or, again neglecting products of small quantities, $a\Delta\theta = \theta\Delta a$. Thus ΔK may be written in the alternate form

$$\Delta K = \frac{\Delta\theta}{a\theta} \tag{1.2}$$

A simple calculation next shows that the strain $\Delta l/l$ in the outermost fibers is given by the relation

$$\frac{\Delta l}{l} = \frac{b}{a + b}\left(\frac{\Delta a}{a} - \frac{\Delta b}{b}\right) \tag{1.3}$$

Consequently the effect of flattening will be appreciable as soon as the ratio $\Delta b/b$ is of the same order of magnitude as $\Delta a/a$.

Detailed analysis shows that the initial curvature of the center line of the tube not only increases the flexibility but has other effects as well. Marked deviations may occur from the linear distribution of stresses over the cross section which holds for a tube with straight axis, and

All the results referred to so far were obtained by use of minimum principles into which approximate expressions were introduced for displacements or stresses. In the present paper the problem is considered instead from a differential equation point of view. This is accomplished by considering bending of a curved tube as a problem of the theory of thin shells. This means that axisymmetrical stress distributions are to be determined in shells of revolution for which the circumferential component of displacement is not axisymmetric.

The differential equations which are appropriate to a problem of this kind were obtained recently by one of the present authors (8). It was subsequently found that, for the special case of tubes with uniform wall thickness and piecewise uniform radius of curvature of the cross section, analogous differential equations had previously been obtained by Tueda (10).

On the basis of the differential equation formulation of the general problem we reexamine in what follows the problem of the tube with uniform circular cross section. Expansion of the solution in trigonometric series leads to an infinite system of linear equations for the coefficients of the series. We find that from this infinite system of equations we may obtain both the results previously obtained by application of the principle of minimum potential energy and by application of the principle of least work. The difference between the two types of results may be looked upon, within the scope of the present approach, as a difference between ways of reducing the infinite system of linear equations to a finite system.

Using trigonometric series, the larger the value of the parameter b^2/ah the larger is the number of terms which must be taken in the series to obtain quantitative results. In what follows, we show that this difficulty can be overcome by asymptotic integration of the differential equations of the problem. A direct check on the accuracy of this procedure is possible because of the overlapping of the ranges of parameter values where series solutions and asymptotic solutions are practical. In the asymptotic range we arrive at simple explicit formulas for all quantities of interest.

Mathematically the problem consists in finding the asymptotic form of a *particular integral* of a nonhomogeneous linear differential equation in which a large parameter multiplies a coefficient function which vanishes for certain values of its argument. Consequently our solution may be considered as an extension of what is often referred to as the W.K.B.J. method for the solution of *homogeneous* equations of this kind.

The method of asymptotic integration is first applied to the problem of the bending of a tube with circular cross section and uniform wall

thickness. However, the same method is also applicable to problems concerning tubes with noncircular cross sections or with cross sections of nonuniform thickness. This is illustrated in the present paper by obtaining simple explicit results for a tube with elliptical cross section and uniform wall thickness. These results for an elliptical tube include those obtained for a tube with circular cross section as a special case.

II. Formulation of the Problem (8)

Considering the curved tube as part of a thin shell of revolution (Fig. 2), let the equation of the middle surface of this shell be written in the following parametric form

$$r = r(\xi), \qquad z = z(\xi) \tag{2.1}$$

Let

$$u = u(\xi), \qquad w = w(\xi) \tag{2.2}$$

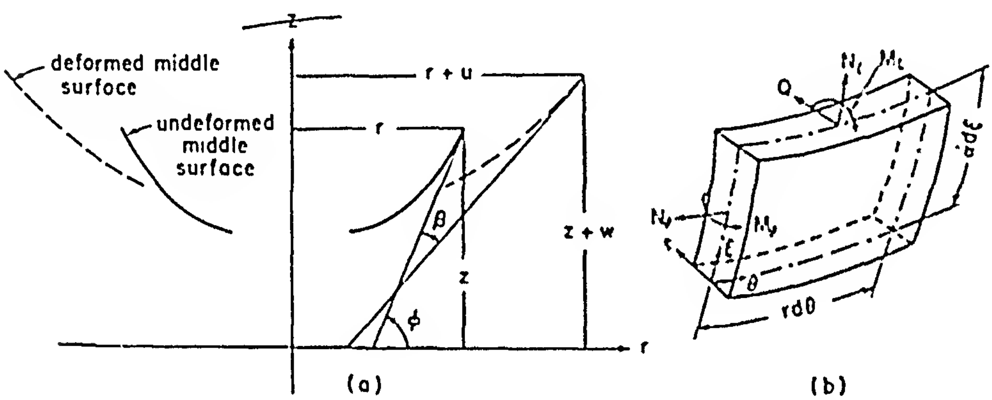


Fig. 2

be the components of displacement in radial and axial directions of points of the middle surface. Let

$$v = kr\theta \tag{2.3}$$

be the circumferential component of displacement, where θ is the polar angle in the plane $z = 0$. Let

$$\beta = \beta(\xi) \tag{2.4}$$

be the change, due to deformation, of the slope angle ϕ of the meridian curves of the shell.

Let N_ξ, N_θ, and Q be stress resultants, and let M_ξ and M_θ be stress couples in accordance with Fig. 2b.

Assuming the absence of surface loads, the stress resultants may all be expressed by means of the conditions of force equilibrium for elements of the shell in the following form,

$$N_\xi = \frac{r'}{\alpha r}\Psi \tag{2.5}$$

$$Q = -\frac{z'}{\alpha r}\Psi \tag{2.6}$$

$$N_\theta = \frac{1}{\alpha}\Psi' \tag{2.7}$$

In equations (2.5) to (2.7) Ψ is a stress function, primes indicate differentiation with respect to ξ, and α is defined by

$$\alpha^2 = r'^2 + z'^2 \tag{2.8}$$

The stress function Ψ may be interpreted as the product of the radial distance r and a horizontal stress resultant $H = N_\xi \cos\phi - Q\sin\phi$.

In addition to the two conditions of force equilibrium, which are accounted for by (2.5) to (2.7), we have a condition of moment equilibrium which may be written in the form

$$(rM_\xi)' - r'M_\theta + z'\Psi = 0 \tag{2.9}$$

Admitting the possibility of elastically orthotropic behavior of the material, the stress resultants and couples are taken to be related to direct and bending strains in the following form

$$\epsilon_{\xi M} = \frac{N_\xi}{C_\xi} - \frac{N_\theta}{C_{\xi\theta}}, \qquad \epsilon_{\theta M} = \frac{N_\theta}{C_\theta} - \frac{N_\xi}{C_{\xi\theta}} \tag{2.10}$$

$$M_\xi = D_\xi\,\kappa_\xi + D_{\xi\theta}\,\kappa_\theta, \qquad M_\theta = D_\theta\,\kappa_\theta + D_{\xi\theta}\,\kappa_\xi \tag{2.11}$$

In equations (2.10) and (2.11) strains are expressed in terms of the displacements u and v and the rotation β as follows.

$$\epsilon_{\xi M} = \frac{u'}{r'} - \frac{z'\beta}{r'}, \qquad \epsilon_{\theta M} = \frac{u}{r} + k \tag{2.12}$$

$$\kappa_\xi = \frac{\beta'}{\alpha}, \qquad \kappa_\theta = \frac{r'\beta}{r\alpha} \tag{2.13}$$

Equations (2.12) imply a compatibility relation of the form

$$(r\epsilon_{\theta M})' - r'\epsilon_{\xi M} = z'\beta + r'k \tag{2.14}$$

One now combines equations (2.9), (2.11), and (2.13) on the one hand and equations (2.14), (2.10), (2.7), and (2.5) on the other hand to obtain the following two simultaneous equations for β and Ψ

$$\beta'' + \frac{(rD_\xi/\alpha)'}{(rD_\xi/\alpha)}\beta' - \left[\left(\frac{r'}{r}\right)^2\frac{D_\theta}{D_\xi} - \frac{(r'D_{\xi\theta}/\alpha)'}{(rD_\xi/\alpha)}\right]\beta + \frac{z'\Psi}{(rD_\xi/\alpha)} = 0 \tag{2.15}$$

$$\Psi'' + \frac{(r/\alpha C_\theta)'}{(r/\alpha C_\theta)}\Psi' - \left[\left(\frac{r'}{r}\right)^2\frac{C_\theta}{C_\xi} + \frac{(r'/\alpha C_{\xi\theta})'}{(r/\alpha C_\theta)}\right]\Psi - \frac{z'\beta}{(r/\alpha C_\theta)} = \frac{r'k}{(r/\alpha C_\theta)} \tag{2.16}$$

When k is set equal to zero, elastic isotropy is assumed, and only edge moments and horizontal edge loads are applied, the foregoing two equations reduce to those given by H. Reissner and E. Meissner for axisymmetrical bending with axisymmetrical displacement.

Having obtained Ψ and β, the components of displacement u and w are determined by means of the following relations.

$$u = r\epsilon_{\theta M} - rk \tag{2.17}$$
$$w = \int (z'\epsilon_{\xi M} - r'\beta)d\xi \tag{2.18}$$

The relation between the constant k and the change of curvature ΔK as defined by equation (1.2) is obtained as follows. From (2.3) $ka\theta = a\Delta\theta$ where a is that value of r for which $\epsilon_{\theta M} = 0$; that is, a is the radius of the neutral (cylindrical) surface. Hence $k = \Delta\theta/\theta$ and hence, in view of (1.2), we have the following relation between the change of curvature ΔK and the constant k.

$$\Delta K = \frac{k}{a} \tag{2.19}$$

Having solved equations (2.15) and (2.16) subject to suitable periodicity conditions for closed tubes, or to suitable boundary conditions for slitted tubes, a relation between the applied bending moment m and the change of curvature ΔK is obtained from the following relation which defines m.

$$m = \int rN_\theta \alpha d\xi - \int z'M_\theta d\xi \tag{2.20}$$

For a tube with closed cross section the integral involving M_θ in (2.20) can be neglected as contributing a term of the same relative order of magnitude as those disregarded in formulating the problem as one of the theory of *thin* shells. For a tube with open cross section, however, both terms should be retained. In this way it is possible to treat, for instance, the problem of pure bending of a ring with narrow rectangular cross section, where the major axis of the cross section makes a prescribed angle with the axis of the ring, as a problem of the theory of conical shells. So far as the authors know no solution of this problem has yet been given.

III. The Equations for Bending of a Tube with Uniform Circular Cross Section

Let b be the radius of the cross section of the tube and a the radius of the center line. The equations of the middle surface of the shell are then (see Fig. 3 on page 106)

$$r = a + b \sin \xi, \qquad z = -b \cos \xi, \qquad \alpha = b \tag{3.1}$$

Let h be the thickness of the walls and assume that the material is isotropic. Then

$$D_\xi = D_\theta = \frac{Eh^3}{12(1-\nu^2)} \equiv D, \quad D_{\xi\theta} = \nu D \tag{3.2}$$

$$C_\xi = C_\theta = Eh \equiv C, \quad C_{\xi\theta} = C/\nu \tag{3.3}$$

where E is the modulus of elasticity of the material and ν is Poisson's ratio.

Introduce a *dimensionless* stress function ψ defined by

$$\psi = \frac{\sqrt{12(1-\nu^2)}}{Eh^2}\,\Psi \tag{3.4}$$

and parameters λ and μ defined by

$$\lambda = \frac{b}{a}, \qquad \mu = \sqrt{12(1-\nu^2)}\,\frac{b^2}{ah} \tag{3.5}$$

Equations (2.15) and (2.16) now assume the following form

$$\beta'' + \frac{\lambda\cos\xi}{1+\lambda\sin\xi}\beta' - \left[\left(\frac{\lambda\cos\xi}{1+\lambda\sin\xi}\right)^2 + \frac{\nu\lambda\sin\xi}{1+\lambda\sin\xi}\right]\beta + \mu\frac{\sin\xi}{1+\lambda\sin\xi}\psi = 0 \tag{3.6}$$

$$\psi'' + \frac{\lambda\cos\xi}{1+\lambda\sin\xi}\psi' - \left[\left(\frac{\lambda\cos\xi}{1+\lambda\sin\xi}\right)^2 - \frac{\nu\lambda\sin\xi}{1+\lambda\sin\xi}\right]\psi - \mu\frac{\sin\xi}{1+\lambda\sin\xi}\beta = \mu k\frac{\cos\xi}{1+\lambda\sin\xi} \tag{3.7}$$

Equations equivalent to these may be found in Tueda's paper (10). They are there integrated by means of power series in $\sin\xi$ which, it appears, is a less natural procedure for closed tubes than the one adopted in what follows.

Before proceeding with the analysis we observe that, by assumption, $h/b \ll 1$ inasmuch as we have restricted attention to the problem of the *thin shell*. According to (3.5) we may write $\mu = \sqrt{12(1-\nu^2)}\,\lambda(b/h)$ so that for a *thin* shell we always have

$$\lambda \ll \mu \tag{3.8}$$

We may then simplify equations (3.6) and (3.7) to

$$\beta'' + \frac{\lambda\cos\xi}{1+\lambda\sin\xi}\beta' + \mu\frac{\sin\xi}{1+\lambda\sin\xi}\psi = 0 \tag{3.9}$$

$$\psi'' + \frac{\lambda\cos\xi}{1+\lambda\sin\xi}\psi' - \mu\frac{\sin\xi}{1+\lambda\sin\xi}\beta = \mu k\frac{\cos\xi}{1+\lambda\sin\xi} \tag{3.10}$$

It may be verified that this form of the two basic equations is obtained directly from (2.15) and (2.16) if a limiting form of orthotropy is assumed, namely,

$$D_\xi = D, \quad D_\theta = 0, \quad D_{\xi\theta} = 0, \quad C_\theta = C, \quad 1/C_\xi = 0, \quad 1/C_{\xi\theta} = 0 \tag{3.11}$$

This means that the bending moments M_θ are disregarded, which is consistent if we disregard them in (2.20), and that the direct strains in the meridional direction are disregarded, which is consistent, since the theory of thin shells disregards transverse shearing strains and the meridional direct strains are here of the same order of magnitude as these shearing strains (see equations 3.12 and 3.13). The stress resultants and couples as defined by (2.5) to (2.7), (2.11) and (2.13) now become in terms of the solutions of equations (3.9) and (3.10)

$$N_\xi = \frac{Eh^2}{a\sqrt{12(1-\nu^2)}} \frac{\cos \xi}{1 + \lambda \sin \xi} \psi \tag{3.12}$$

$$Q = -\frac{Eh^2}{a\sqrt{12(1-\nu^2)}} \frac{\sin \xi}{1 + \lambda \sin \xi} \psi \tag{3.13}$$

$$N_\theta = \frac{Eh^2}{b\sqrt{12(1-\nu^2)}} \psi' \tag{3.14}$$

$$M_\xi = \frac{Eh^3}{b12(1-\nu^2)} \beta' \tag{3.15}$$

$$M_\theta = \nu M_\xi \tag{3.16}$$

An expression for the radial component of displacement u follows from (2.17), if in this we express $\epsilon_{\theta M}$ in terms of ψ, in the form

$$u = [(b/\mu)\psi' - ak](1 + \lambda \sin \xi) \tag{3.17}$$

This may be transformed by means of the differential equation (3.10) into

$$u = b\int \beta \sin \xi \, d\xi - ak \tag{3.17a}$$

The axial component of displacement w is, according to (2.18),

$$w = -b\int \beta \cos \xi \, d\xi \tag{3.18}$$

In reducing (2.18) to (3.18) we have again taken account of the fact that $\lambda \ll \mu$.

Finally the relation (2.20) for the applied bending moment m becomes

$$m = -\frac{Eb^3h}{a\mu} \int_0^{2\pi} \psi \cos \xi \, d\xi \tag{3.19}$$

In general the parameter $\lambda = b/a$ in equations (3.9) and (3.10) is quite small compared with unity and may for practical purposes be set

equal to zero. This is what has been done in fact by von Kármán (6), Lorenz (7), and Beskin (2). Some sample calculations concerning the effect of finite λ have been carried out by Karl (5) and these confirm the suitability of assuming $\lambda = 0$. Setting $\lambda = 0$ means the same for the present problem as disregarding the difference between hyperbolic and linear bending stress distribution means for solid beams with curved axis.

Under the assumption that $\lambda = 0$ equations (3.9) and (3.10) become

$$\beta'' + \mu \sin \xi \, \psi = 0 \tag{3.20}$$
$$\psi'' - \mu \sin \xi \, \beta = \mu k \cos \xi \tag{3.21}$$

The integration of this system of equations will be considered next.

IV. Trigonometric Series Solution for the Tube with Uniform Circular Cross Section

For the simplified differential equations (3.20) and (3.21) the solution possesses properties of symmetry and antisymmetry with respect to the points $\xi = 0, \pi, \pm\pi/2$ of such nature that suitable expansions are of the form

$$\beta = -\mu k \rho \sum_{1}^{\infty} a_{2n} \sin 2n\xi \tag{4.1}$$

$$\psi = -\mu k \rho \left[\cos \xi + \sum_{2}^{\infty} b_{2n-1} \cos (2n - 1)\xi \right] \tag{4.2}$$

The significance of the factor ρ may be seen by introducing (4.2) into the expression (3.19) for the applied moment m. This gives

$$m = EI \frac{k}{a} \rho \tag{4.3}$$

where

$$I = \pi b^3 h \tag{4.4}$$

is the moment of inertia of the cross section of the tube. Since k/a is the change in curvature it follows that ρ may be designated as a *rigidity factor* such that ρEI is the *effective* stiffness of the tube. When $a = \infty$ and therewith $\mu = 0$ we have $\rho = 1$. For all other values of μ the rigidity factor ρ will be less than 1.

Let σ_{SM} be the maximum fiber stress which occurs in the straight tube, i.e.,

$$\sigma_{SM} = \frac{mb}{I} \tag{4.5}$$

Combination of equations (4.1) and (4.2) with equations (3.12) through (3.15) then gives the following expressions for direct and bending stresses.

$$\sigma_{\theta D} = \frac{N_\theta}{h} = \left[\sin\xi + \sum_2^\infty (2n-1)b_{2n-1}\sin(2n-1)\xi\right]\sigma_{SM} \quad (4.6)$$

$$\sigma_{\xi D} = \frac{N_\xi}{h} = -\frac{\lambda}{2}\left[1 + \sum_1^\infty (b_{2n-1} + b_{2n+1})\cos 2n\xi\right]\sigma_{SM} \quad (4.7)$$

$$\tau_0 = \frac{3Q}{2h} = -\frac{3\lambda}{4}\left[\sum_1^\infty (b_{2n-1} - b_{2n+1})\sin 2n\xi\right]\sigma_{SM} \quad (4.8)$$

$$\sigma_{\xi B} = \frac{6M_\xi}{h^2} = -\sqrt{\frac{12}{1-\nu^2}}\left[\sum_1^\infty n a_{2n}\cos 2n\xi\right]\sigma_{SM} \quad (4.9)$$

Let the amount of flattening in the horizontal (radial) direction and the amount of bulging in the vertical (axial) direction of the cross sections of the tube be δ_H and δ_V, respectively. It then follows from equations (3.18) and (3.17) that

$$\delta_V = 2w(0) = -\mu a\frac{\sigma_{SM}}{E}\sum_1^\infty \frac{4na_{2n}}{(2n-1)(2n+1)} \quad (4.10)$$

$$\delta_H = u\left(-\frac{\pi}{2}\right) - u\left(\frac{\pi}{2}\right) = \mu a\frac{\sigma_{SM}}{E}\sum_1^\infty (-1)^{n+1}\frac{4na_{2n}}{(2n-1)(2n+1)} \quad (4.11)$$

Having expressed stresses and displacements in terms of the coefficients a_{2n} and b_{2n-1} it is next necessary to determine these coefficients and also the value of the rigidity factor ρ. To this end equations (4.1) and (4.2) are introduced into the differential equations (3.20) and (3.21), which leads to the following infinite set of linear equations.

$$(2n-1)^2 b_{2n-1} - \tfrac{1}{2}\mu(a_{2n-2} - a_{2n}) = \begin{cases} 1/\rho, & n = 1 \\ 0, & n = 2,3,\ \cdots \end{cases} \quad (4.12)$$

$$(2n)^2 a_{2n} - \tfrac{1}{2}\mu(b_{2n-1} - b_{2n+1}) = 0 \quad (4.13)$$

In equations (4.12) and (4.13) n assumes all positive integer values, the coefficient b_1 is unity, and coefficients with negative or zero indices have the value zero. Thus the first of each of equations (4.12) and (4.13) are

$$1 + \frac{1}{2}\mu a_2 = \frac{1}{\rho} \quad (4.12a)$$

$$4a_2 - \tfrac{1}{2}\mu(1 - b_3) = 0 \quad (4.13a)$$

Assuming convergence of the series (4.1) and (4.2), approximate values of the coefficients may be obtained by neglecting all coefficients except the first few in the series. The approximation will become closer as more and more coefficients are taken into account.

Let us set $b_{2m+1} = 0$ so that we are taking account of m terms in the series for the stress function ψ. We may then proceed in one of two ways.

One procedure is to reduce equations (4.12) and (4.13) to a single system of equations for the coefficients b_{2n-1} by introducing

$$a_{2n} = \frac{\mu}{2}\frac{b_{2n-1} - b_{2n+1}}{(2n)^2}$$

from (4.13) into (4.12). This means that we have

$$a_{2m} = \frac{\mu}{2}\frac{b_{2m-1}}{(2m)^2} \neq 0$$

and thus we are taking account of m terms also in the series for β.

A second procedure consists in introducing

$$b_{2n-1} = \frac{\mu}{2}\frac{a_{2n-2} - a_{2n}}{(2n-1)^2}$$

from (4.12) into (4.13). In order to have now as many equations as unknowns we must set

$$a_{2m} = 0$$

and thus we are taking account of only $m - 1$ terms in the series for β.

We find that by means of the first procedure, which retains an equal number of terms in the series for both ψ and β, we reproduce the results which can be obtained by an application of Castigliano's theorem of least work as used by Lorenz (7), Karl (5), and Beskin (2). Application of the second procedure, which retains one less term in the series for β than in the series for ψ, leads, on the other hand, to the results which can also be obtained by application of the principle of minimum potential energy as used by von Kármán (6).

In what follows we list the results of the various approximations, thereby extending and completing results previously obtained by earlier authors.

1. No β-Terms, One ψ-Term Retained

This leads to no correction due to the curvature of the center line of the tube as equations (4.12a) and (4.13a) give, with $a_2 = b_3 = 0$, the simple result that $\rho = 1$.

2. One β-Term, One ψ-Term Retained

With $b_3 = 0$ in equation (4.13a), we have $a_2 = \frac{1}{8}\mu$, and therewith from (4.12a) the following expression for the rigidity factor ρ,

$$\rho = \frac{16}{16 + \mu^2} \tag{4.14}$$

Equations (4.1) and (4.2) now become

$$\beta = -\tfrac{1}{8}k\rho\mu^2 \sin 2\xi \tag{4.15}$$
$$\psi = -k\rho\mu \cos \xi \tag{4.16}$$

Direct and bending stresses as given by (4.6) and (4.9) become

$$\sigma_{\theta D} = \sigma_{SM} \sin \xi \tag{4.17}$$
$$\sigma_{\xi B} = -\frac{3\mu}{2\sqrt{12(1-\nu^2)}}\sigma_{SM} \cos 2\xi \tag{4.18}$$

Expressions for horizontal flattening and vertical bulging follow from (4.10) and (4.11) in the form

$$\delta_H = -\delta_V = \frac{a\mu^2}{6}\frac{\sigma_{SM}}{E} \tag{4.19}$$

The foregoing approximation was first obtained by Lorenz (7), who did not carry his work beyond this stage. More extensive results by Karl (5) and Beskin (2) include the above as a zeroth approximation except that these authors do not explicitly determine the amount of flattening and bulging.

It is noteworthy that with the above approximation no account is taken of the effect of flattening on the distribution of the primary direct stress $\sigma_{\theta D}$.

3. One β-Term, Two ψ-Terms Retained

The following formulas are obtained.

$$\rho = \frac{16 + \frac{1}{9}\mu^2}{16 + \frac{10}{9}\mu^2} \tag{4.14a}$$
$$\beta = -\tfrac{1}{8}k\rho\mu^2(1 - \tfrac{1}{144}\mu^2) \sin 2\xi \tag{4.15a}$$
$$\psi = -k\rho\mu\left(\cos \xi + \frac{\mu^2}{144 + \mu^2}\cos 3\xi\right) \tag{4.16a}$$
$$\frac{\sigma_{\theta D}}{\sigma_{SM}} = \sin \xi + \frac{3\mu^2}{144 + \mu^2}\sin 3\xi \tag{4.17a}$$
$$\frac{\sigma_{\xi B}}{\sigma_{SM}} = -\frac{3\mu}{2\sqrt{12(1-\nu^2)}}\left(1 - \frac{\mu^2}{144 + \mu^2}\right)\cos 2\xi \tag{4.18a}$$
$$\delta_H = -\delta_V = \frac{a\mu^2}{6}\left(1 - \frac{\mu^2}{144 + \mu^2}\right)\frac{\sigma_{SM}}{E} \tag{4.19a}$$

The above results correspond to what von Kármán (6) called his first approximation. Explicit formulas for $\sigma_{\xi B}$ and δ_H and δ_V for this approximation are given here for the first time.

Of particular interest is the fact that now the maximum value of the direct fiber stress $\sigma_{\theta D}$ does not necessarily occur at the point $\xi = \pi/2$, which is farthest away from the neutral axis. The following results

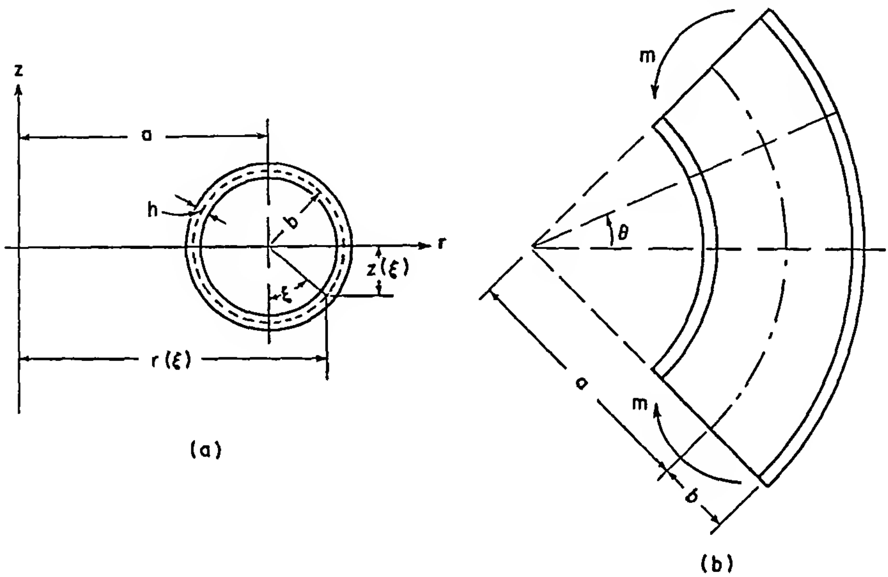


FIG. 3

are found for the maximum value, $\sigma_{\theta DM}$, of $\sigma_{\theta D}$ on the basis of equation (4.17a)

$$\begin{aligned} \sigma_{\theta DM} &= \sigma_{\theta D}\left(\frac{\pi}{2}\right) = \left(1 - \frac{3\mu^2}{144 + \mu^2}\right)\sigma_{SM}, \quad \mu < 2.353 \\ \sigma_{\theta DM} &= \sigma_{\theta D}(\xi_M) = \frac{1}{9\mu}\frac{(144 + 10\mu^2)^{3/2}}{(144 + \mu^2)}\sigma_{SM}, \quad \mu > 2.353 \end{aligned} \tag{4.20}$$

where

$$\xi_M = \sin^{-1}\sqrt{\frac{4}{\mu^2} + \frac{40}{144}} \tag{4.21}$$

This result may be compared with the maximum value $\sigma_{\xi BM}$ of the bending stress $\sigma_{\xi B}$ which occurs at $\xi = 0$ and which is given by

$$\frac{\sigma_{\xi BM}}{\sigma_{SM}} = -\frac{3\mu}{2\sqrt{12(1-\nu^2)}}\left(1 - \frac{\mu^2}{144 + \mu^2}\right) \tag{4.22}$$

Comparison of equations (4.20) and (4.22) indicates that for sufficiently small values of μ the bending stress $\sigma_{\xi B}$ is smaller than the direct stresses

$\sigma_{\theta D}$ but that this situation reverses itself as μ increases beyond the value 2, approximately (see Fig. 7).

4. Two β-Terms, Two ψ-Terms Retained

We now find the following results

$$\rho = \frac{16 + \frac{5}{36}\mu^2}{16 + \frac{41}{36}\mu^2 + \frac{1}{576}\mu^4} \tag{4.14b}$$

$$\beta = -\frac{1}{8}k\rho\mu^2\left(\frac{16 + \frac{1}{36}\mu^2}{16 + \frac{5}{36}\mu^2}\sin 2\xi + \frac{\frac{1}{36}\mu^2}{16 + \frac{5}{36}\mu^2}\sin 4\xi\right) \tag{4.15b}$$

$$\psi = -k\rho\mu\left(\cos\xi + \frac{\mu^2}{144 + \frac{5}{4}\mu^2}\cos 3\xi\right) \tag{4.16b}$$

$$\frac{\sigma_{\theta D}}{\sigma_{SM}} = \sin\xi + \frac{3\mu^2}{144 + \frac{5}{4}\mu^2}\sin 3\xi \tag{4.17b}$$

$$\frac{\sigma_{\xi B}}{\sigma_{SM}} = -\frac{3\mu}{2\sqrt{12(1-\nu^2)}}\left[\left(1 - \frac{\mu^2}{144 + \frac{5}{4}\mu^2}\right)\cos 2\xi + \frac{\frac{1}{3}\mu^2}{144 + \frac{5}{4}\mu^2}\cos 4\xi\right] \tag{4.18b}$$

$$\begin{aligned} \delta_H &= \frac{a\mu^2}{6}\left(1 - \frac{\frac{11}{16}\mu^2}{144 + \frac{5}{4}\mu^2}\right)\frac{\sigma_{SM}}{E} \\ -\delta_V &= \frac{a\mu^2}{6}\left(1 - \frac{\frac{9}{16}\mu^2}{144 + \frac{5}{4}\mu^2}\right)\frac{\sigma_{SM}}{E} \end{aligned} \tag{4.19b}$$

Equation (4.14b) for ρ corresponds to Karl's (5) first approximation for this quantity by means of the theorem of least work. This author also proves that the exact value of ρ is greater than the value given by (4.14b) and less than the value given by von Kármán's first approximation (4.14a).

Concerning the maximum fiber stress $\sigma_{\theta DM}$ we now find the following results.

$$\begin{aligned} \sigma_{\theta DM} &= \sigma_{\theta D}\left(\frac{\pi}{2}\right) = \left(1 - \frac{3\mu^2}{144 + \frac{5}{4}\mu^2}\right)\sigma_{SM}, \quad \mu < 2.365 \\ \sigma_{\theta DM} &= \sigma_{\theta D}(\xi_M) = \frac{1}{9\mu}\frac{(144 + \frac{41}{4}\mu^2)^{3/2}}{(144 + \frac{5}{4}\mu^2)}\sigma_{SM}, \quad \mu > 2.365 \end{aligned} \tag{4.23}$$

where

$$\xi_M = \sin^{-1}\sqrt{\frac{4}{\mu^2} + \frac{41}{144}} \tag{4.24}$$

The results are seen to be quite similar to those in equations (4.20) and (4.21).

The maximum values of the bending stress $\sigma_{\xi B}$ follow from (4.18b) in the form

$$\frac{\sigma_{\xi BM}}{\sigma_{SM}} = -\frac{3\mu}{2\sqrt{12(1-\nu^2)}}\left(1 - \frac{\frac{1}{3}\mu^2}{144 + \frac{5}{4}\mu^2}\right) \tag{4.25}$$

Comparison of (4.25) and (4.22) shows that the step from the preceding to the present approximation has more effect on the values of $\sigma_{\xi BM}$ than it has on those of $\sigma_{\theta DM}$. This is to be expected as $\sigma_{\xi B}$ is given directly by an expression which has now two terms instead of the one before.

We finally note that, for the first time with this approximation the amount of radial flattening does not coincide with the amount of axial bulging as given by equations (4.19b).

We could continue the discussion along these lines but evidently explicit results become progressively more complex.[1] Instead we shall next show that an asymptotic integration procedure makes it unnecessary to go farther than we have done here with the trigonometric series solution.

Before doing this, however, it may be of some interest to list two expressions for the rigidity factor ρ in the form of continued fractions. These continued fraction expansions are obtained from the recurrence relations (4.12) and (4.13) after these relations are written either in terms of the coefficients a_{2n} or b_{2n-1}. The possibility of these expansions is due to the fact that the recurrence relations are *three-term* relations. The results are as follows.

$$\rho = 1 - \cfrac{3^2\mu^2}{4\cdot 1^2\cdot 2^2\cdot 3^2 + (1^2+3^2)\mu^2 - \cfrac{5^2\mu^4}{4\cdot 3^2\cdot 4^2\cdot 5^2 + (3^2+5^2)\mu^2 - \cdots}}$$
$$- \frac{(2n-3)^2(2n+1)^2\mu^4}{4(2n-1)^2(2n)^2(2n+1)^2 + [(2n-1)^2+(2n+1)^2]\mu^2 - \cdots} \tag{4.26}$$

or

$$\rho = \cfrac{16}{16+\mu^2 - \cfrac{4^2\mu^4}{4\cdot(2\cdot 3\cdot 4)^2 + (2^2+4^2)\mu^2 - \cdots}}$$
$$- \frac{(2n-2)^2(2n+2)^2\mu^4}{4(2n)^2(2n+1)^2(2n+2)^2 + [(2n)^2+(2n+2)^2]\mu^2 - \cdots} \tag{4.27}$$

If one retains one, two, or three fractions on the right of (4.26), one obtains von Kármán's first, second, or third approximation for the rigidity factor ρ.

Similarly equation (4.27) leads to the successive approximations for ρ which have been obtained by Karl by the application of Castigliano's theorem.

[1] Numerical calculations which correspond to a retention of *six* terms in both the β and the ψ-series may be found in the paper by Beskin (2) which has been referred to previously.

V. Asymptotic Solution for the Tube with Uniform Circular Cross Section

In order to obtain solutions of the differential equations (3.20) and (3.21) which for sufficiently large values of the parameter μ are more convenient than the series solutions (4.1) and (4.2) we proceed as follows. We first reduce the two real equations (3.20) and (3.21) to one complex equation by introducing a function Z defined by

$$Z = \beta + i\psi \tag{5.1}$$

The differential equation for Z is

$$Z'' - i\mu \sin \xi \, Z = i\mu k \cos \xi \tag{5.2}$$

Because of the symmetry of the state of deformation with respect to the radial and axial lines through the cross section of the tube, it is sufficient to determine the function Z in the interval $0 \leq \xi \leq \pi/2$. We have at $\xi = 0$ the symmetry conditions of horizontal tangent and of vanishing longitudinal fiber stress which expressed in terms of β and ψ read as follows

$$\beta(0) = 0, \qquad \psi'(0) = 0 \tag{5.3}$$

In order to obtain conditions at $\xi = \pi/2$ it is observed, on the basis of the trigonometric series solutions, that the range of appreciable stresses $\sigma_{\theta B}$ and $\sigma_{\xi B}$ becomes a narrower and narrower region in the neighborhood of $\xi = 0$ as the value of the parameter μ increases. This suggests application of the method of asymptotic integration with respect to the now large parameter μ. Restricting attention for the moment to the neighborhood of $\xi = \pi/2$ it is apparent that a particular integral of (5.2) may be obtained by an expansion in inverse powers of μ. We may express this fact in a form which is appropriate for the present purposes by writing

$$Z = -k \cot \xi + O\left(\frac{1}{\mu}\right) \tag{5.4}$$

Evidently this solution ceases to be significant in the neighborhood of $\xi = 0$. However, when ξ is small we have $\sin \xi \approx \xi$, $\cos \xi \approx 1$ and equation (5.2) may then be approximated by

$$Z'' - i\mu\xi \, Z = i\mu k \tag{5.5}$$

It remains to interpolate between the solution of (5.5) and the expression (5.4) and this we do in the following manner. We introduce

a new variable x defined by

$$x = \mu^{1/3} \sin \xi \tag{5.6}$$

and a new function T defined by

$$Z = i\mu^{1/3} k \cos \xi \, T(x) \tag{5.7}$$

The function Z of (5.7) will satisfy the boundary condition (5.4) provided we have, as $x \to \infty$,

$$T(x) \sim \frac{i}{x} \tag{5.8}$$

For values of ξ in the neighborhood of $\xi = 0$ we have from (5.7) and (5.6)

$$Z \approx i\mu^{1/3} k T(\mu^{1/3} \xi) \tag{5.9}$$

This expression will satisfy equation (5.5) provided $T(x)$ satisfies the differential equation

$$\frac{d^2T(x)}{dx^2} - ix\, T(x) = 1 \tag{5.10}$$

From (5.1) and (5.7) the relation

$$\psi - i\beta = \mu^{1/3} k \cos \xi \, T(x) \tag{5.11}$$

follows. Writing

$$T(x) = T_r(x) + iT_i(x) \tag{5.12}$$

(5.11) reduces to

$$\psi = \mu^{1/3} k \cos \xi \, T_r(x) \tag{5.13}$$

$$\beta = -\mu^{1/3} k \cos \xi \, T_i(x) \tag{5.14}$$

The boundary conditions (5.3) become the following conditions for the real and imaginary parts of the function $T(x)$,

$$T_i(0) = 0, \qquad T_r'(0) = 0 \tag{5.15}$$

The boundary condition (5.8) may be written in the form

$$\lim_{x \to \infty} xT_i(x) = 1, \qquad \lim_{x \to \infty} xT_r(x) = 0 \tag{5.16}$$

A solution of the basic differential equation (5.10) which satisfies the limit conditions (5.16) can be expressed in terms of a suitable Lommel function as follows (11)

$$T(x) = -\tfrac{2}{3}(ix)^{1/2} S_{0,1/3}[\tfrac{2}{3}(ix)^{3/2}] \tag{5.17}$$

It happens that this solution also satisfies the boundary conditions (5.15) so that no solution of equation (5.10) with right-hand side set

equal to zero has to be superimposed. This is seen on the basis of the following representations of $T(x)$ as defined by (5.17)

$$
\begin{aligned}
T_r(x) &= s_r(x) + \gamma[h_{1,r}(ix) + \sqrt{3}\,h_{1,i}(ix) + h_{2,r}(ix) - \sqrt{3}\,h_{2,i}(ix)] \\
T_i(x) &= s_i(x) + \gamma[-\sqrt{3}\,h_{1,r}(ix) + h_{1,i}(ix) + \sqrt{3}\,h_{2,r}(ix) + h_{2,i}(ix)] \\
T'_r(x) &= s'_r(x) + \gamma[\sqrt{3}\,h'_{1,r}(ix) - h'_{1,i}(ix) - \sqrt{3}\,h'_{2,r}(ix) - h'_{2,i}(ix)] \\
T'_i(x) &= s'_i(x) + \gamma[h'_{1,r}(ix) + \sqrt{3}\,h'_{1,i}(ix) + h'_{2,r}(ix) - \sqrt{3}\,h'_{2,i}(ix)]
\end{aligned}
\tag{5.18}
$$

In equations (5.18) h_1 and h_2 are modified Hankel functions of order one-third which have recently been tabulated (3), γ is a constant given by

$$\gamma = \left(\frac{\pi}{6\sqrt{3}}\right)\left(\frac{3}{2}\right)^{\frac{1}{3}} = 0.346 \cdots \tag{5.19}$$

and s_r and s_i are power series of the form

$$
\begin{aligned}
s_r(x) &= \sum_0^\infty \frac{(-1)^n\Gamma(\frac{1}{3})\Gamma(\frac{2}{3})}{\Gamma(\frac{1}{3}+2n+1)\Gamma(\frac{2}{3}+2n+1)}\frac{x^{6n+2}}{3^{4n+2}} \\
s_i(x) &= \sum_0^\infty \frac{(-1)^n\Gamma(\frac{1}{3})\Gamma(\frac{2}{3})}{\Gamma(\frac{1}{3}+2n+2)\Gamma(\frac{2}{3}+2n+2)}\frac{x^{6n+5}}{3^{4n+4}}
\end{aligned}
\tag{5.20}
$$

For large values of x the following asymptotic expansions hold

$$
\begin{aligned}
T_r(x) &= \frac{1\cdot 2}{x^4} - \frac{(1\cdot 4\cdot 7)(2\cdot 5\cdot 8)}{x^{10}} + \cdots \\
T_i(x) &= \frac{1}{x} - \frac{(1\cdot 4)(2\cdot 5)}{x^7} + \cdots
\end{aligned}
\tag{5.21}
$$

For the purpose of calculating values of the functions T and T' by numerical integration of the differential equation (5.10), it is noted that the representation (5.18) to (5.20) implies satisfaction of the following additional initial conditions

$$T_r(0) = -1.288, \qquad T_i'(0) = 0.939 \tag{5.22}$$

Figure 4 contains graphs of the values of the functions T_r, T_i, T_r', T_i' which are needed for application of the theory.

We may now apply the foregoing considerations in determining the quantities which are of interest in the tube bending problem. We begin by obtaining an expression for the rigidity factor ρ as defined by equation

(4.3). Combination of (4.3), (4.4), and (3.19) gives

$$\rho = -\frac{1}{\pi\mu k}\int_0^{2\pi} \psi \cos \xi \, d\xi \tag{5.23}$$

In (5.23) we introduce (5.13) to obtain

$$\rho = -\frac{4}{\pi\mu^{2/3}}\int_0^{\pi/2} T_r(x) \cos^2 \xi \, d\xi \tag{5.24}$$

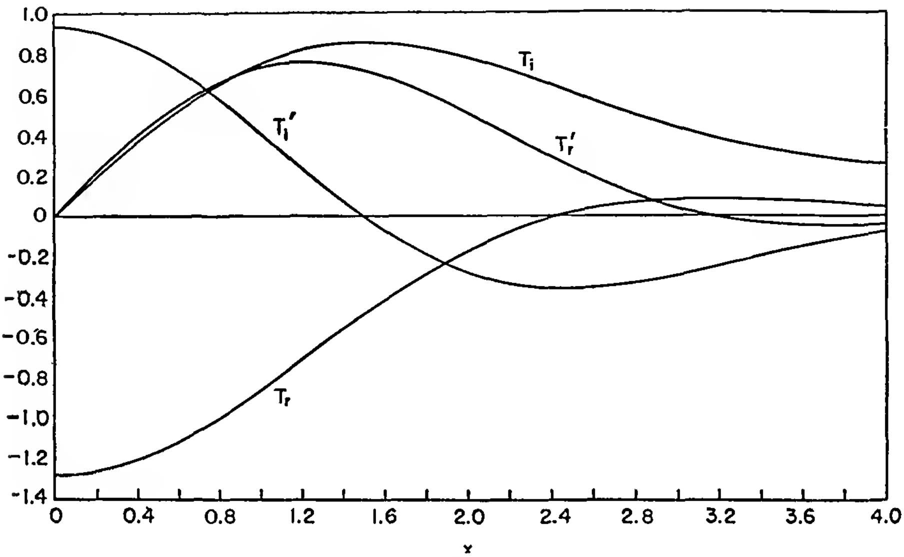


FIG. 4. Values of $T_r(x)$, $T_i(x)$ and their derivatives.

Introducing x as the variable of integration by means of (5.6), this becomes

$$\rho = -\frac{4}{\pi\mu}\int_0^{\mu^{1/3}} \sqrt{1 - \left(\frac{x}{\mu^{1/3}}\right)^2}\; T_r(x)dx \tag{5.25}$$

In view of the behavior of the function T_r the main contribution to the value of the integral comes from the neighborhood of $x = 0$ so that we may replace (5.25) by the simpler relation

$$\rho = -\frac{4}{\pi\mu}\int_0^{\infty} T_r(x)dx \tag{5.26}$$

The value of the integral in (5.26) has been determined by numerical integration and within three-figure accuracy it is found that

$$\int_0^{\infty} T_r(x)dx = -\frac{\pi}{2} \tag{5.27}$$

We thus have the following simple asymptotic expression for the rigidity factor ρ,

$$\rho = \frac{2}{\mu} \tag{5.28}$$

Equation (5.28) is to be compared with equations (4.14) which give ρ for sufficiently small values of μ. By more refined analysis it should be possible to establish explicit bounds for the error associated with the use of equation (5.28). We content ourselves here with a comparison

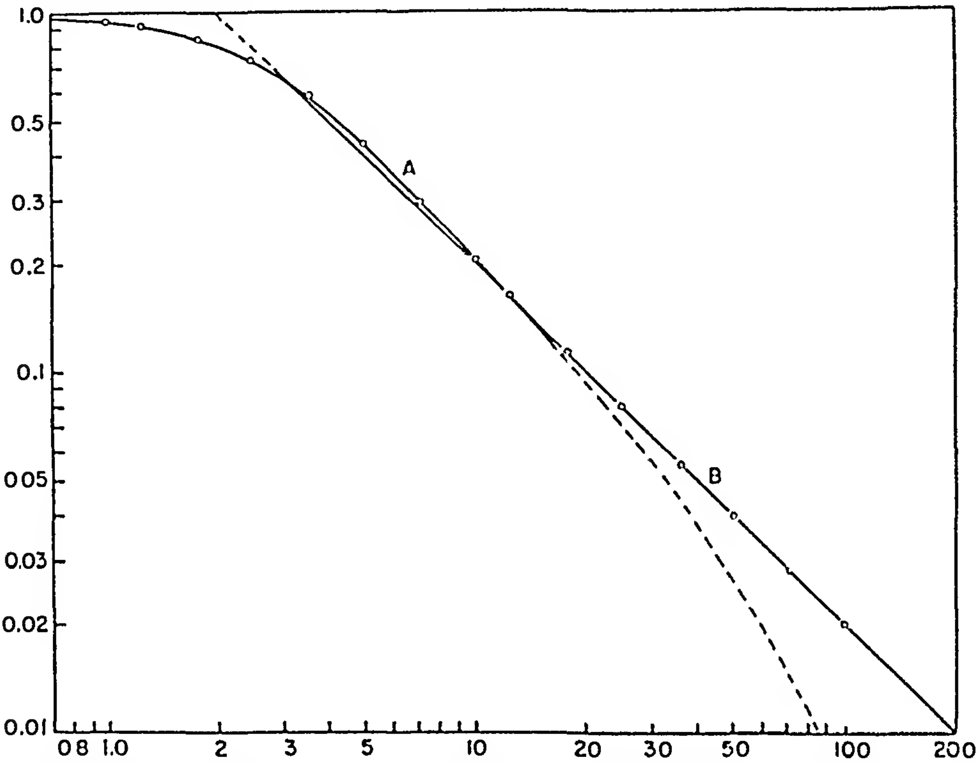


FIG. 5. Rigidity factor ρ as function of the parameter $\mu = \sqrt{12(1 - \nu^2)}\,(b^2/ah)$. Curve A represents results according to equation (4.14b). Curve B represents asymptotic results. Circles represent numerical values tabulated in (2).

with the numerical computations of Beskin (2) based on trigonometric series for β and ψ containing six terms each. This comparison shows that equation (5.28) is in error by less than 2 per cent for μ greater than 10 and that the percentage error decreases as μ increases. For values of μ less than 10, equation (4.14b) (two β-terms, two ψ-terms retained) is adequate. The two equations (4.14b) and (5.28) then cover the entire range of μ-values with an error of less than 2 per cent.

Equations (4.14b) and (5.28) are plotted in Fig. 5 as well as Beskin's results. It is remarkable that the asymptotic form (5.28) is quite accurate even for values of μ as small as 3.

Turning next to the determination of the values of the circumferential fiber stress $\sigma_{\theta D}$ we have, in view of (3.14) and (5.13),

$$\sigma_{\theta D} = \frac{N_\theta}{h} = \frac{Eh}{b\sqrt{12(1-\nu^2)}} \frac{d}{d\xi} [\mu^{1/3} k \cos \xi T_r(x)] \tag{5.29}$$

As we know from the trends of the trigonometric series solutions that for sufficiently large values of μ the values of the stress $\sigma_{\theta D}$ are negligible everywhere except for values of ξ which are near zero, we may simplify equation (5.29) by setting in it $\cos \xi = 1$ and $\xi = \mu^{-1/3} x$. This gives

$$\sigma_{\theta D} = \frac{Eh\mu^{2/3}k}{b\sqrt{12(1-\nu^2)}} \frac{dT_r(x)}{dx}, \quad \xi \ll \frac{\pi}{2} \tag{5.30}$$

According to (4.3) and (4.5)

$$k = \frac{am}{EI\rho} = \frac{\sigma_{SM}}{E} \frac{a}{b\rho} \tag{5.31}$$

Introduction of k from (5.31) and ρ from (5.28) with μ as defined in (3.5) leads finally to the following expression for $\sigma_{\theta D}$,

$$\frac{\sigma_{\theta D}}{\sigma_{SM}} = \frac{1}{2} \mu^{2/3} \frac{dT_r(x)}{dx}, \quad x = \mu^{1/3} \xi, \quad \xi \ll \frac{\pi}{2} \tag{5.32}$$

From equation (5.32) we may determine for what value $\xi = \xi_M$ the stress $\sigma_{\theta D}$ assumes its maximum value $\sigma_{\theta DM}$ and what the magnitude of $\sigma_{\theta DM}$ is. A maximum of $\sigma_{\theta D}$ occurs when $T_r''(x) = 0$, or, according to the differential equation (5.10), when $xT_i(x) = 1$. The smallest value of x for which this equation holds is $x = 1.225$. Consequently

$$\xi_M = \frac{1.225}{\mu^{1/3}} \tag{5.33}$$

which shows that as μ increases $\sigma_{\theta D}$ becomes largest at a section nearer and nearer to the neutral axis of the cross section of the tube.

Since $T_r'(1.225) = 0.753$ it follows from (5.32) that $\sigma_{\theta DM}$ is given for sufficiently large values of μ by the following simple formula,

$$\sigma_{\theta DM} = 0.377 \mu^{2/3} \sigma_{SM} \tag{5.34}$$

Again comparing the asymptotic formulas obtained here with Beskin's numerical calculations it is found that equation (5.33) for ξ_M is accurate within 2 per cent for μ greater than 5, and that equation (5.34) for $\sigma_{\theta DM}$ is accurate within 2 per cent for μ greater than 12.5. On the basis of Fig. 7 it is to be noted that the asymptotic formula for $\sigma_{\theta DM}$ is qualitatively correct for values of μ as small as $\mu = 4$.

In Fig. 6 we have plotted the values of ξ_M in accordance with formulas (4.24) and (5.33). Figure 7 shows the values of $\sigma_{\theta DM}$ as given by (4.23) and (5.34). It is recalled that (4.23) and (4.24) were obtained by retaining two terms in each of the series for β and ψ.

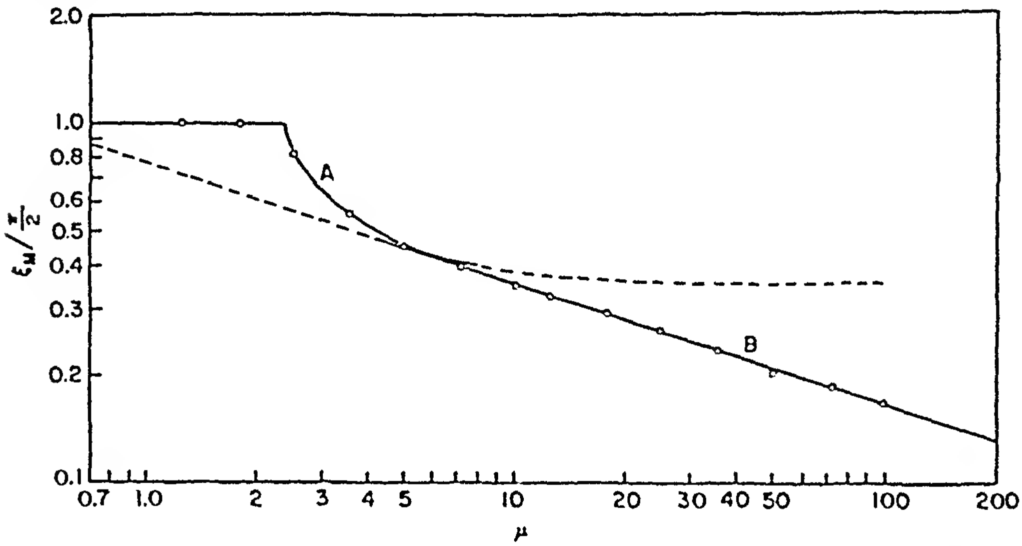


FIG. 6. Angle ξ_M for which maximum circumferential stress $\sigma_{\theta DM}$ occurs plotted as a function of μ. Curve A represents results according to equation (4.24). Curve B represents asymptotic results. Circles represent values tabulated in (2).

It remains to obtain an asymptotic formula for the bending stress $\sigma_{\xi B}$. From (3.15) and (4.9) it is seen that

$$\sigma_{\xi B} = \frac{Eh}{2b(1 - \nu^2)} \frac{d\beta}{d\xi} \tag{5.35}$$

and, with (5.14)

$$\sigma_{\xi B} = -\frac{Eh\mu^{1/3}k}{2b(1 - \nu^2)} \frac{d}{d\xi} [\cos \xi \; T_i(x)] \tag{5.36}$$

The stress $\sigma_{\xi B}$ assumes its maximum $\sigma_{\xi BM}$ for $\xi = 0$. Proceeding in the same manner as in going from (5.30) to (5.32) we find

$$\frac{\sigma_{\xi BM}}{\sigma_{SM}} = -\frac{\sqrt{3}}{2} \frac{\mu^{2/3}}{\sqrt{1 - \nu^2}} \left(\frac{dT_i}{dx}\right)_{x=0} \tag{5.37}$$

With $T_i'(0) = 0.939$ this formula may be written in the form

$$\sigma_{\xi BM} = -0.813(1 - \nu^2)^{-1/2}\mu^{2/3}\sigma_{SM} \tag{5.38}$$

A comparison of (5.38) and (5.34) shows that for sufficiently large values of μ, say for $\mu > 10$, $\sigma_{\xi BM}$ and $\sigma_{\theta DM}$ are proportional to each other and

that the wall bending stress $\sigma_{\xi BM}$ is more than twice as large as the maximum value $\sigma_{\theta DM}$ of the longitudinal fiber stress.

The values of $\sigma_{\xi BM}$ given by (5.38) do not agree as closely with the numerical results of Beskin as do the values of $\sigma_{\theta DM}$ given by (5.34). For $\mu = 10$ the difference between Beskin's value and that obtained from (5.38) is 7.4 per cent. This difference diminishes as μ increases

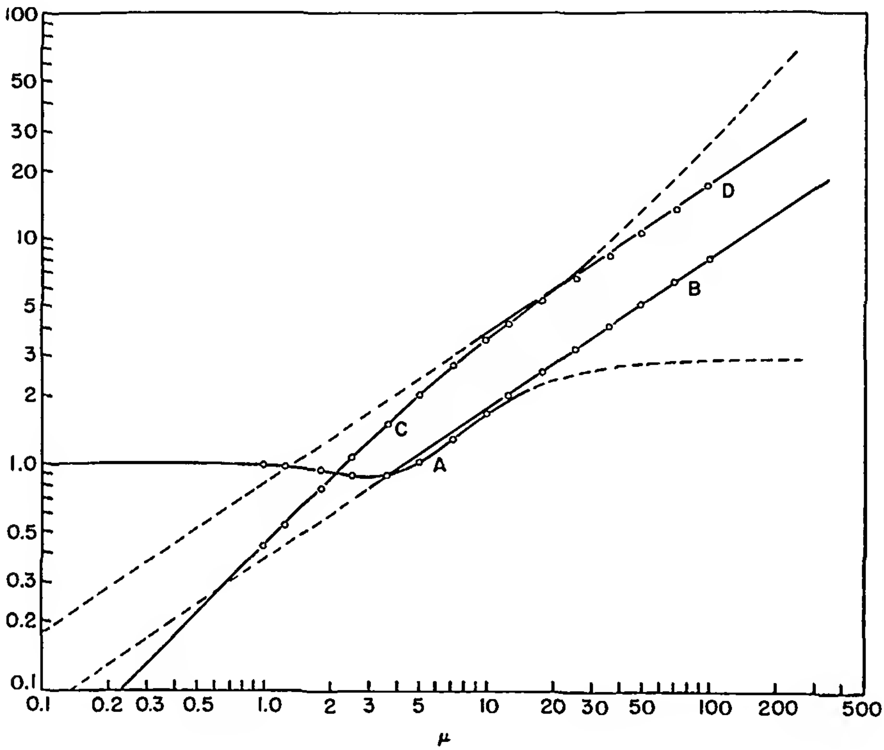


FIG. 7. Maximum circumferential direct stress $\sigma_{\theta DM}$ and maximum meridional bending stress $\sigma_{\xi BM}$ plotted as functions of μ. Curve A represents $\sigma_{\theta DM}/\sigma_{SM}$ as given by equation (4.23). Curve B represents $\sigma_{\theta DM}/\sigma_{SM}$ as given by asymptotic formula (5.34). Curve C represents $(1 - \nu^2)^{1/2}\sigma_{\xi BM}/\sigma_{SM}$ as given by equation (4.18b). Curve D represents $(1 - \nu^2)^{1/2}\sigma_{\xi BM}/\sigma_{SM}$ as given by asymptotic formula (5.38). Circles represent the values of the corresponding quantities as tabulated in (2).

and is less than 2 per cent for $\mu = 100$. These differences are remarkable since the asymptotic values for $\sigma_{\xi BM}$ should be as accurate as the corresponding values of $\sigma_{\theta DM}$.

We finally list in this section an expression for the amount of flattening δ_H of the tube. From (3.17a) it follows that

$$\delta_H = u\left(-\frac{\pi}{2}\right) - u\left(\frac{\pi}{2}\right) = -2b \int_0^{\pi/2} \beta \sin \xi \, d\xi \tag{5.39}$$

Into this formula is introduced β from equation (5.14) and $d\xi$ is expressed in terms of dx by means of equation (5.6) to obtain

$$\delta_H = \frac{2bk}{\mu^{1/2}} \int_0^{\mu^{1/2}} xT_i(x)dx \tag{5.40}$$

Since we have $xT_i = 1 - T_r''$, according to the differential equation (5.10), and $T_r'(0) = 0$ according to (5.15), it follows that

$$\delta_H = 2bk\left[1 - \frac{T_r'(\mu^{1/2})}{\mu^{1/2}}\right] \tag{5.41}$$

or, within the accuracy of the present calculations,

$$\delta_H = 2bk \tag{5.41a}$$

With k from (5.31) this may also be written in the form

$$\frac{\delta_H}{a} = \mu\frac{\sigma_{SM}}{E} \tag{5.41b}$$

If equation (5.41b) is compared with the corresponding expression (4.19b) valid for smaller values of μ, it is seen that the quadratic variation with μ tends to become linear as μ increases.

An alternate form of (5.41b) is

$$\frac{\delta_H}{h} = \sqrt{12(1 - \nu^2)}\left(\frac{b}{h}\right)^2 \frac{\sigma_{SM}}{E} \tag{5.41c}$$

This is of interest as indicating that in the range of sufficiently large values of μ the amount of flattening for given bending moment is independent of the curvature of the center line of the tube.

Corresponding to equation (5.40) for the flattening of the tube we obtain from equation (3.18) the following expression for the amount of vertical bulging

$$\delta_V = -2bk \int_0^{\mu^{1/2}} \sqrt{1 - \left(\frac{x}{\mu^{1/2}}\right)^2}\, T_i(x)dx \tag{5.42}$$

We content ourselves with observing that δ_V as given by (5.42) is of the same order of magnitude in μ as is δ_H, but do not attempt here to further evaluate (5.42).

VI. The Equations for Bending of a Tube with Uniform Elliptical Cross Section

The parametric equations of the middle surface of the shell may be taken in the form (Fig. 8)

$$r = a + b\sin\xi, \qquad z = -c\cos\xi \tag{6.1}$$

The differential equations (2.15) and (2.16) assume the following form for an elliptical tube of isotropic material

$$\beta'' + \frac{\lambda(1 - e^2)\cos\xi - e^2 \sin\xi\cos\xi}{(1 + \lambda\sin\xi)(1 - e^2\cos^2\xi)}\beta' - \left[\left(\frac{\lambda\cos\xi}{1 + \lambda\sin\xi}\right)^2 + \frac{\nu\lambda\sin\xi}{(1 + \lambda\sin\xi)(1 - e^2\cos^2\xi)}\right]\beta + \frac{\mu\sin\xi\sqrt{1 - e^2\cos^2\xi}}{\sqrt{1 - e^2}\,(1 + \lambda\sin\xi)}\psi = 0 \tag{6.2}$$

$$\psi'' + \frac{\lambda(1 - e^2)\cos\xi - e^2\sin\xi\cos\xi}{(1 + \lambda\sin\xi)(1 + e^2\cos^2\xi)}\psi' - \left[\left(\frac{\lambda\cos\xi}{1 + \lambda\sin\xi}\right)^2 - \frac{\nu\lambda\sin\xi}{(1 + \lambda\sin\xi)(1 - e^2\cos^2\xi)}\right]\psi - \frac{\mu\sin\xi\sqrt{1 - e^2\cos^2\xi}}{\sqrt{1 - e^2}\,(1 + \lambda\sin\xi)}\beta = \frac{\mu k\cos\xi\sqrt{1 - e^2\cos^2\xi}}{1 + \lambda\sin\xi} \tag{6.3}$$

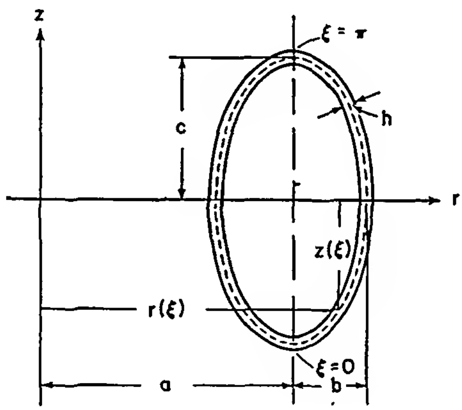


FIG. 8

In (6.2) and (6.3) the stress function ψ is defined as before by equations (3.4) and (2.5) to (2.7). The parameter μ is now of the form

$$\mu = \sqrt{12(1 - \nu^2)}\,\frac{bc}{ah} \tag{6.4}$$

while $\lambda = b/a$ as before. The parameter e is the *eccentricity* of the ellipse; that is,

$$e^2 = 1 - \left(\frac{b}{c}\right)^2 \tag{6.5}$$

VII. Asymptotic Solution for the Tube with Uniform Elliptical Cross Section

For small values of μ the problem has recently been treated by Huber (4) by means of the principle of minimum potential energy. In what

follows we apply the method of asymptotic integration to obtain explicit results for sufficiently large values of μ. If we assume that b/c is of order unity, that is if we exclude extremely narrow tubes then, for $\mu \gg 1$, equations (6.2) and (6.3) may be replaced by the following simpler equations.

$$\beta'' + \mu \sin\xi \sqrt{(1 - c^2\cos^2\xi)/(1 - c^2)}\,\psi = 0 \tag{6.6}$$
$$\psi'' - \mu \sin\xi \sqrt{(1 - c^2\cos^2\xi)/(1 - c^2)}\,\beta = \mu k \cos\xi \sqrt{1 - c^2\cos^2\xi} \tag{6.7}$$

When $c = 0$ these two equations reduce, as they should, to equations (3.20) and (3.21) for the tube with circular cross section.

For the function $Z = \beta + i\psi$ we have now the complex differential equation

$$Z'' - i\mu \sin\xi \sqrt{(1 - c^2\cos^2\xi)/(1 - c^2)}\, Z = i\mu k \cos\xi \sqrt{1 - c^2\cos^2\xi} \tag{6.8}$$

which corresponds to equation (5.2) for the circular tube.

We now proceed in complete analogy to what has been done in section V for the case $c = 0$ and obtain the following approximate solution of equation (6.8) which satisfies appropriate boundary conditions for $\xi = 0$ and $\xi = \pi/2$,

$$Z = i\mu^{1/3} k \cos\xi \sqrt{1 - c^2\cos^2\xi}\; T(x) \tag{6.9}$$

In this equation $T(x)$ is defined by (5.17) and the variable x is now given by

$$x = \mu^{1/3} \sin\xi \sqrt{(1 - c^2\cos^2\xi)/(1 - c^2)} \tag{6.10}$$

An expression for the rigidity factor ρ follows from (2.20) and (4.3) according to which

$$\rho EI \frac{k}{a} = \int_0^{2\pi} r\alpha N_\theta \, d\xi \tag{6.11}$$

With N_θ from (2.7) and Ψ from (3.4) there results from (6.11), after integration by parts, the following expression for ρ.

$$\rho = -\frac{4hcb^2}{I} \int_0^{\pi/2} \frac{\psi}{\mu k} \cos\xi \, d\xi \tag{6.12}$$

The moment of inertia I of the tube cross section is given by

$$\begin{aligned} I &= \int_0^{2\pi} (r - a)^2 h\alpha \, d\xi \\ &= 4hcb^2 \int_0^{\pi/2} \sin^2\xi \sqrt{1 - e^2\cos^2\xi}\, d\xi \\ &= 4hcb^2 J(e) \end{aligned} \tag{6.13}$$

The integral $J(e)$ may be expressed in terms of complete elliptic integrals of the first and second kind as follows.

$$3e^2J(e) = (1 + e^2)E(e) - (1 - e^2)K(e) \tag{6.14}$$

For $0 \leq e \leq 1$ the quantity $J(e)$ varies within the narrow range

$$J(0) = \frac{\pi}{4}, \qquad J(1) = \frac{2}{3} \tag{6.15}$$

Combination of (6.12) and (6.13) gives

$$\rho = \frac{-1}{J(e)} \int_0^{\pi/2} \frac{\psi}{\mu k} \cos \xi \, d\xi \tag{6.16}$$

In the range of values of μ for which the asymptotic solution (6.9) applies equation (6.16) becomes, with $\psi = \text{Imag.}\ \{Z\}$,

$$\rho = \frac{-1}{J(e)} \int_0^{\pi/2} \mu^{-2/3}\, T_r(x) \cos^2 \xi \sqrt{1 - e^2 \cos^2 \xi}\, d\xi \tag{6.17}$$

When $e = 0$ equation (6.17) reduces to (5.24) as it must do. If now $d\xi$ is expressed in terms of dx by means of (6.10), equation (6.17) is changed to

$$\rho = \frac{-\sqrt{1 - e^2}}{J(e)\mu} \int_0^{\mu^{1/3}/(1-e^2)^{1/2}} T_r(x) \frac{1 - e^2 \cos^2 \xi}{1 - e^2 \cos 2\xi} \cos \xi \, dx \tag{6.18}$$

and this, in view of the character of the solution for large x, may be replaced by

$$\rho = \frac{-\sqrt{1 - e^2}}{J(e)\mu} \int_0^{\infty} T_r(x) dx = \frac{\pi \sqrt{1 - e^2}}{2\mu J(e)} \tag{6.19}$$

If we write equation (6.19) in the form

$$\rho = \frac{\pi}{4J(e)} \frac{ah}{\sqrt{3(1 - \nu^2)}\, c^2} \tag{6.19a}$$

and observe that $\pi/4J(e)$ varies by less than 20 per cent as $e^2 = 1 - (b/c)^2$ varies from zero to unity, we may conclude that, for sufficiently large values of μ (μ greater than 5, say), the rigidity factor of the elliptical tube is nearly the same as the corresponding factor for the circular tube with cross-sectional radius equal to the major axis of the elliptical section, provided the major axis of the elliptical section is parallel to the axis of revolution.[2]

[2] Since this account was written we have carried out some calculations for the tube with elliptical cross section, using the trigonometric series method. We find that

We may again determine the location and magnitude of the maximum circumferential fiber stress $\sigma_{\theta DM}$. We find that

$$\sigma_{\theta D} = \frac{N_\theta}{h} = \frac{Eh}{\alpha\sqrt{12(1-\nu^2)}}\,\psi'(\xi) \tag{6.20}$$

is equivalent to

$$\sigma_{\theta D} = \frac{\mu^{2/3} kEh}{\sqrt{12(1-\nu^2)}\,c}\,T_r'(x) \tag{6.20a}$$

in the range of applicability of the asymptotic solution. With $k/a = m/\rho EI$ and $\sigma_{SM} = mb/I$ (6.20a) may be written in the form

$$\sigma_{\theta D} = \frac{\sigma_{SM}}{\rho\mu^{1/3}}\,T_r'(x) \tag{6.20b}$$

where ρ is to be taken from (6.19). As for the circular tube we find from (6.20b) that the maximum value, $\sigma_{\theta DM}$, of $\sigma_{\theta D}$ occurs for $x = 1.225$. Again assuming $\cos\xi \approx 1$ and $\sin\xi \approx \xi$ it follows that the position of the maximum is given by $\xi_M = 1.225/\mu^{1/3}$ or

$$\xi_M = \frac{1.225}{\sqrt[6]{12(1-\nu^2)}}\cdot\sqrt[3]{\frac{ah}{bc}} \tag{6.21}$$

Equation (6.21) gives directly the distance $r_M - a$ from the neutral axis of the most highly stressed circumferential fiber in units of the semi-axis b.

With $T_r'(1.225) = 0.753$ we obtain from (6.20b) for the maximum of $\sigma_{\theta D}$,

$$\frac{\sigma_{\theta DM}}{\sigma_{SM}} = \frac{0.753}{\rho\mu^{1/3}} = 0.377\,\frac{4J(c)}{\pi}\,\frac{\mu^{2/3}}{\sqrt{1-c^2}} \tag{6.22}$$

Written in more explicit form (6.22) becomes

$$\frac{\sigma_{\theta DM}}{\sigma_{SM}} = 0.377\,\frac{4J(c)}{\pi}\,[12(1-\nu^2)]^{1/3}\,\frac{c^{1/3}}{b^{1/3}a^{2/3}h^{2/3}} \tag{6.22a}$$

We finally determine the maximum value of the bending stress $\sigma_{\xi B}$ in the range of validity of the asymptotic theory. We have in view of (2.11) and (2.13)

$$\sigma_{\xi B} = \frac{6M_\xi}{h^2} = \frac{Eh}{2(1-\nu^2)}\,\frac{\beta'}{\alpha} \tag{6.23}$$

---

while somewhat larger values of μ are required than for the tube with circular cross section for the same degree of accuracy of the asymptotic solution, the asymptic results are within a few per cent of the series results as soon as μ has a value of about 10.

As β' is largest for $\xi = 0$ it follows from (6.23) that

$$\sigma_{\xi BM} = \frac{Eh}{2(1 - \nu^2)c} \frac{\beta'(0)}{\sqrt{1 - e^2}} = - \frac{Eh\mu^{2/3}k}{2(1 - \nu^2)c} T_i'(0) \tag{6.24}$$

With $T_i'(0) = 0.939$ and performing the same steps as in going from (6.20a) to (6.20b) we find

$$\frac{\sigma_{\xi BM}}{\sigma_{BM}} = - \frac{0.939\sqrt{3}}{\sqrt{1 - \nu^2}} \frac{1}{\rho\mu^{1/3}} = - \frac{0.813}{\sqrt{1 - \nu^2}} \frac{4J(e)}{\pi} \frac{\mu^{2/3}}{\sqrt{1 - e^2}} \tag{6.25}$$

A comparison of (6.25) and (6.22) indicates the fact that the ratio of maximum bending to maximum direct stress is constant for values of μ which are large enough for the asymptotic theory to be applied and that the value of this ratio, $\sigma_{\xi BM}/\sigma_{\theta DM} = 2.16/(1 - \nu^2)^{1/2}$ is independent of the ratio of the semi-axes b and c of the cross section of the tube.

References

1. Bantlin, A., Formänderung und Beanspruchung federnder Ausgleichsröhren, *Z. Ver. deut. Ing.*, **54**, 43–49 (1910).
2. Beskin, L., Bending of curved thin tubes, *J. Applied Mechanics*, **12**, A1–A7 (1945).
3. Harvard Computation Laboratory, Annals of, vol. 2, *Tables of the Modified Hankel Functions of Order One-Third and of Their Derivatives*, Cambridge, Mass., Harvard University Press, 1945.
4. Huber, M. T., Pure bending of curved thin-walled tubes of elliptic cross section, (In Polish) *Arch. Mech. Stos.* **1**, 1–22 (1949). See also *Proc. VIIth Int. Congr. Appl. Mech.*, **1**, 322 (1949)
5. Karl, H., Biegung gekrümmter, dünnwandiger Rohre, *Z. angew. Math. Mech.*, **23**, 331–345, (1943).
6. von Kármán, Th., Uber die Formänderung dünnwandiger Rohre, insbesondere federnder Ausgleichsrohre, *Z. Ver. deut. Ing.*, **55**, 1889–1895 (1911).
7. Lorenz, H., Die Biegung krummer Rohre, *Physik. Z.*, **13**, 768–774 (1912).
8. Reissner, E., On bending of curved thin-walled tubes, *Proc. Natl. Acad. Sci. U.S.*, **35**, 204–208 (1949).
9. Timoshenko, S., Bending stresses in curved tubes of rectangular cross section, *Trans. Am. Soc. Mech. Engs.*, **45**, 135–140 (1923).
10. Tueda, M., Mathematical theories of Bourdon pressure tubes and bending of curved pipes, *Mem. Coll. Eng., Kyoto Imp. Univ.*, **8**, 102–115 (1934); **9**, 132–152 (1936).
11. Watson, G. N., *A Treatise on the Theory of Bessel Functions*, New York, Macmillan, 1944.

Recent Developments in Inverse and Semi-Inverse Methods in the Mechanics of Continua[1]

By P. F. Neményi

Naval Research Laboratory, Washington, D.C.

CONTENTS

I. Introduction 123
II. Motion of Incompressible Inviscid Fluids 124
III. Flow of a Viscous Incompressible Fluid 126
IV. Gas Flows 130
V. Elastostatics 136
VI. Plasticity 141
VII. Significance and Heuristic Value of the Inverse and Semi-Inverse Methods of Mechanics 145
References 148

I. Introduction

We shall call inverse an investigation of a partial differential equation of physics if in it the boundary conditions (or certain other supplementary conditions) are not prescribed at the outset. Instead, the solution is defined by the differential equation, and certain additional analytical, geometrical, kinematical, or physical properties of the field. In the semi-inverse method some of the boundary conditions are prescribed at the outset, whereas others are left open and obtained indirectly through certain simplifying assumptions concerning the properties of the field. A classic example for this is St. Venant's theory of the elastic rod in which only stress systems are considered which leave the mantle surface of a cylindrical rod free from loads, while the exact distribution of the forces on the two bases is not prescribed in advance. There is, of course, no sharp line of demarcation between the inverse and the semi-inverse procedure. Obviously our subject overlaps considerably with that of "exact" or closed solutions of the equations of the mechanics of continua, although they are by no means identical. The inverse and semi-inverse method in most cases reduces a system of differential equations in three independent variables to a system having only two, or one, independent variables which may, or may not, admit an exact solution in closed form.

[1] Sponsored by the Office of Naval Research. A major part of this paper was written while the author was connected with the Naval Ordnance Laboratory, White Oak, Md.

On the other hand, exact solutions may come from a direct approach. Investigations based upon an inverse approach do not necessarily lead to new solutions. Often the result is negative; that is, the nonexistence of certain types of solutions is established.

Inverse and semi-inverse investigations have been made in all fields of the mechanics of continua. The variety and volume of the results obtained in the various fields is very great; a survey aiming at completeness would require a whole book and a bibliography containing about 200 or more items. In the present paper, therefore, merely characteristic, and, it is hoped, representative examples will be given from the various fields, with the aim to elucidate the nature, the value, and the potentialities of this approach. The methodic questions are dealt with more systematically in the concluding parts of the paper.

Although methodic viewpoints were decisive, personal preferences of the present writer were also influential in the selection of the material presented. Part of it originates in the writer's own research work, or in his joint work with Prim, and includes some as yet unpublished results.

II. Motion of Incompressible Inviscid Fluids

The first example of the use of inverse methods in fluid dynamics is to be found in Euler's earliest paper on hydrodynamics [Novi Comm. Acad. Petr. **6** (1756–57), 271–311 (1761); this paper is earliest in respect to composition, though not in respect to publication]. Having proved that what is now called the velocity potential of an irrotational motion of an inviscid fluid must satisfy the equation $\nabla^2\Phi = 0$, Euler noted that any such function Φ yields a possible fluid motion. He then calculated and discussed the polynomial solutions up to the fifth degree, thus obtaining the first nontrivial exact solutions for any theory of the dynamics of continua.

In the early development of the dynamics of inviscid, incompressible fluids the method of singularities, especially the use of concentrated sources, sinks, doublets, and vortices, played a considerable role. In the early work of Rankine (1) and Kelvin the singularity method was a semi-inverse method, because the boundary conditions could be prescribed at the outset of the investigation only very vaguely. From this approach there gradually developed a method for the direct solution of a wide variety of boundary-value problems of the potential function. I do not intend to discuss the singularity methods of the theory of potential flow, because they go beyond the scope of the present paper; in fact taking into account important new developments, they would require a separate report comparable in length to the present one.

Another inquiry characteristic of the inverse approach is this: Find all irrotational, steady-flow patterns of an incompressible fluid for which the velocity magnitude is constant along each streamline. If we restrict attention to plane irrotational flow, only the circulating flow (induced by a single vortex point) and the uniform, rectilinear flow fulfill the conditions; this follows immediately from simple relations of the theory of functions of a complex variable. However, in space the question is not without difficulty, and was not solved completely before 1937. In that year von Laue conjectured, and Hamel (2) proved, that the only flow in space satisfying this condition is that which is obtained by superimposing upon a circulating flow a uniform flow perpendicular to its plane. The streamlines of this combined flow are obviously ordinary coaxial helices, which are orthogonal to a family of congruent, helicoidal, minimum surfaces.

A remarkable generalization of this problem is the following. Which "doubly laminar" flow fields, that is, fields the velocity of which is the product of a scalar and an irrotational vector, have constant velocity magnitude along each streamline? The complete answer was given, after preliminary work of Castoldi (3), by Prim (3a) in an as yet unpublished memoir.[2]

For general, that is, non doubly laminar rotational flow, the condition of constant velocity magnitude along streamlines is not yet known, except for the case of straight streamlines. A very complete discussion of these flows without acceleration was given by Merlin (4) in 1938.

Flows of a constant velocity magnitude along each streamline have an exceptional position from the standpoint of the uniqueness of the solution. It is quite obvious that *any* velocity field of an incompressible, inviscid fluid in the absence of body forces (and this is assumed throughout this group of investigations) can be multiplied by any constant scalar factor and still satisfy the Euler equations. Quite recently, however, Gilbarg (5) has shown for the plane, and Prim (6) for the general spatial case, that flow patterns admitting a constant velocity magnitude along each streamline are the only ones which are consistent with essentially different velocity distributions, that is, which admit a nonconstant scalar multiplicator in the velocity field.

Opposite, so to speak, to the doubly laminar flow is a flow for which $\mathbf{v} \times \text{curl } \mathbf{v} = 0$, where ($\mathbf{v}$ is the velocity vector). This broad and interesting flow type was discussed in a classic paper by Beltrami (7), who gave unsteady as well as steady flow examples. The following spatial example showing cyclical symmetry may be quoted:

[2] Prim's result is valid also for nonhomogeneous incompressible, and for a broad class of homogeneous compressible fluids.

$$v_x = T_2 \cos 2Ty + T_3 \sin 2Tz$$
$$v_y = T_3 \cos 2Tz + T_1 \sin 2Tx$$
$$v_z = T_1 \cos 2Tx + T_2 \sin 2Ty$$

T_1, T_2, T_3, T being functions of the time t. If we choose T_1 and $T_2 = 0$, we obtain a flow pattern the streamlines of which are at any instant straight lines, but whose direction varies with z and t. This extremely simple case is a special case also of the following flow included in Beltrami's discussion:

$$v_x = \frac{dF_1}{d\zeta} e^{iZ} + \frac{dF_2}{d\zeta} e^{-iZ}$$
$$v_y = i\left[\frac{dF_1}{d\zeta} e^{iZ} - \frac{dF_2}{d\zeta} e^{-iZ}\right],$$

F_1 and F_2 are conjugate complex functions of the variable $\zeta = x + iy$ and of t, while Z is a function of the real variables z, t. This example is (in Berker's (11) terminology) a "pseudo-plane flow of the first kind," that is, a flow which is everywhere parallel to a certain plane, but which varies with the coordinate perpendicular to this plane. Beltrami has shown that plane, irrotational, incompressible flow patterns varying with the parameter z always combine to a flow satisfying the Beltrami condition $\mathbf{v} \times \text{curl } \mathbf{v} = 0$, provided that $\partial v/\partial z = 0$ everywhere. If we omit this condition, we obviously have a pseudo-plane flow of the first kind, the vorticity of which is in the flow plane, but not necessarily in the same direction as $\mathbf{v}$. This generalization of one of the various ideas suggested by Beltrami's paper was discussed with aid of the theory of complex functions by Massotti (8).

III. Flow of a Viscous Incompressible Fluid

More extensive use has been made of inverse and semi-inverse methods in this field than in any other. Nevertheless I will give only little space to it, because three systematic surveys exist already of the "inverse," "exact," and "closed" solutions of the Navier-Stokes equations. First came Nöther's (9) presentation of this subject in the Handbuch der physikalischen und technischen Mechanik, (1930); then Rosenblatt's (10) survey, which in its last section contains some essentially new results (1935); and finally, the recapitulation of known exact solutions in the introductory chapter of Berker's memoir (11) on this subject (1936).

The first major investigation in the field was due to Jeffery (a hydrodynamicist who later became well known for his investigations in the dynamics of nonspherical gravel). In two papers (12,13), published in 1915, Jeffery asked the following questions concerning steady, two-

dimensional flow in absence of body forces: Which are the rotational flows of a viscous fluid, the streamlines of which coincide with those of an irrotational flow? Which are the rotational flows of a viscous fluid, the lines of equal vorticity (isocurls) of which coincide with the streamlines of an irrotational flow? If we call ψ the streamfunction of the plane viscous flow in question and H a harmonic function of the location, the two problems can be formulated in the following way:

$$\psi = F(H)$$

and

$$\nabla^2\psi = F(H)$$

(∇^2 being the Laplacian).

After the preliminary work of Jeffery, Hamel gave (14) a complete solution of the first problem. He showed that H must be the real part either of $A \ln (z - \alpha)$ or of Az (A and α are arbitrary complex constants); that is, rotational, steady, viscous flow can take place in streamlines which admit also an irrotational flow only if they are logarithmic spirals, or their degenerate cases, including parallel straight lines.

For his second question Jeffery found the complete answer: only concentric circles or parallel straight lines are admissible here as streamlines.

Before discussing more radically different problems, we list three direct generalizations of the Jeffery-Hamel problem:

$$\psi = F(H) + c\bar{H} \quad \text{(Oseen, 15)}$$
$$\psi = \bar{H}^n F(H) + G(H) \quad \text{(Rosenblatt, 10)}$$
$$\psi = F(H) + G(\bar{H}) \quad \text{(Berker, 11)}$$

where H and $\bar{H}$ are conjugate harmonic functions of x and y; n is a real constant.

Another closely related question is the one discussed by Görtler and K. Wieghardt (15a): For which plane steady-flow fields of a viscous incompressible fluid, which is under the influence of a conservative force field, does the field of viscous forces possess a nonconstant scalar potential? Surprisingly, the only answer to this question is the Poiseuille flow, that is a straight flow with a parabolic velocity distribution.

For the axially symmetric case, the problem corresponding to Rosenblatt's problem of plane flow has been solved by Berker (after preliminary work by Witoszynski and Szymanski) for $n = 1$. The solution for axially symmetric flow corresponding to Jeffery's second problem of plane flow has, to my knowledge, not yet been found.

Berker's work with steady flow includes also solutions for pseudo-plane and pseudo axially symmetric flows. A pseudo-plane flow of the

first kind is defined, as already mentioned by a velocity field

$$v_x(x,y,z), \qquad v_y(x,y,z), \quad v_z = 0$$

while a pseudo-plane flow of the second kind is one in which all three velocity components are different from 0, but depend only upon x and y. Analogous is the definition of the two kinds of pseudo axially symmetric flow. Berker succeeded in generalizing certain specific plane and axially symmetric solutions to obtain pseudo-plane and pseudo axially symmetric ones.

More important is Berker's method for obtaining from any plane flow, steady or unsteady, a family of solutions involving two arbitrary functions $a(t)$ and $b(t)$ of the time. If $\psi(x,y,t)$ is the given streamfunction, the generalized streamfunction is

$$\psi_1 = a'y - b'x - \frac{\omega}{2}[(x-a)^2 + (y-b)^2]$$
$$+ \psi[(x-a)\cos\omega t + (y-b)\sin\omega t, -(x-a)\sin\omega t + (y-b)\cos\omega t, t]$$

where ω is an arbitrary constant. Berker shows that this method, which is based upon a moving coordinate system, can be extended to certain cases of three-dimensional motion.

As to essentially new nonsteady solutions, one of the simplest nontrivial family of them was obtained by Bateman (16) on the basis of the following assumption

$$\psi = F(x + \alpha t)y + G(x + \alpha t)$$

α being a constant.

Interesting from the standpoint of studying the rate at which a viscous flow, left to its own resources, will lose its velocity is Taylor's (17) assumption

$$\psi(x,y,t) = e^{\nu kt}F(x,y)$$

where ν is viscosity and k an arbitrary constant. It follows immediately from the Navier-Stokes equation for the plane flow that F must then satisfy the equation of membrane vibration

$$\nabla^2 F = kF$$

Obviously, in this solution the streamlines are isocurls. They are also path lines.

As a generalization of this solution, as well as of his own earlier results for steady flow, Kampé de Fériet (18) gave the complete answer to the following question: Find all plane flows satisfying the Navier-Stokes equations for which, in absence of mass forces, the vorticity is

constant along each streamline. Kampé de Fériet proved that this is possible only in the following cases: (1) if the vorticity is constant throughout the field: (2) if the streamlines are concentric circles or parallel straight lines; and (3) for the Taylor flows.

Omitting the unsteady solutions given for plane flow by Oseen and others, and for axially symmetric flow by Berker, we want to mention investigations of viscous, nonsteady flows, which satisfy Beltrami's condition $\mathbf{v} \times \operatorname{curl} \mathbf{v} = 0$. Caldonazzo was the first to study such flows, and Trkal (19) gave a remarkable generalization of some of Caldonazzo's results. In purely formal analogy with Taylor's assumption, Trkal sets:

$$\mathbf{v} = \mathbf{A}(x,y,z)e^{-\nu k^2 t}$$

If now $\operatorname{curl} \mathbf{A} = k\mathbf{A}$, then obviously the Beltrami condition is satisfied for $\mathbf{v}$. Trkal shows that in this case,

$$\nabla^2\mathbf{A} = k^2\mathbf{A}$$

Here again the streamlines are path lines; that is, the streamline pattern is steady. In Berker's investigations, on the other hand, such flow fields are often sought for which the velocity field at every moment is consistent with a steady state of flow, and hence for which the vortex field is steady.

A new, and possibly fruitful, viewpoint was introduced into the theory of viscous flow fields by Ballabh, who in a series of papers (19a,19b) inquires under what conditions two viscous flow fields, superposed, yield again a viscous flow field. One form of the conditions found for homogeneous viscous fluids is that the vorticity of both given fields should satisfy the equation of heat conduction.

While all these are strict solutions of the Navier-Stokes equations, Doucet has set himself an essentially different problem. He inquires into the mathematical nature of the solutions of the differential equations of J. Boussinesq for the time averaged velocity vector $\bar{\mathbf{v}}$ of a statistically steady turbulent flow of an incompressible liquid. These equations (which, if α, the eddy viscosity, is taken to be a constant, can at best have an approximate validity) are:

$$\operatorname{curl} [\alpha\nabla^2\bar{\mathbf{v}} + \operatorname{curl} \bar{\mathbf{v}} \times \nabla^2\bar{\mathbf{v}}] = 0$$

(and $\operatorname{div} \bar{\mathbf{v}} = 0$), $\nabla^2\bar{\mathbf{v}}$ being the Laplacian of $\bar{\mathbf{v}}$

$$(\nabla^2\bar{\mathbf{v}} = \mathbf{i}\nabla^2\bar{v}_x + \mathbf{j}\nabla^2\bar{v}_y + \mathbf{k}\nabla^2\bar{v}_z)$$

In order to obtain families of solutions, Doucet decomposes $\bar{\mathbf{v}}$ in an irrotational term $\operatorname{grad} \Theta$ and a solenoidal term $\mathbf{u}$, and proceeds to obtain solutions for the following cases:

1. $\mathbf{u}$ is a function of a scalar $\phi(x,y,z)$; with special reference to the sub-case $\phi = \Theta$; and
2. curl $\mathbf{u} = \mathbf{u}$.

IV. Gas Flows

Throughout the following discussions we shall use the expression "perfect gas" to denote a nonviscous compressible fluid which is subject to a relation between pressure p and density ρ of the form

$$\frac{p}{p_0} = \left(\frac{\rho}{\rho_0}\right)^{\alpha}$$

In the last few decades the dynamics of perfect gases has made very great progress, and a substantial part of this is due to the inverse and semi-inverse methods.

In the case of plane irrotational flow let us first recall the solution due to Prandtl and Meyer (1908) (20), (sometimes called "simple wave" solution). This family of solutions can be obtained as answer to following question: Given a one-parameter system of straight lines; find all gas flows for which the velocity vector is constant along each of these straight lines, but varies from one line to the other. The answer can be expressed purely in terms of the angle δ formed by the straight lines with a fixed direction:

$$v^2{}_t = v^2{}_{\min} \cos^2 \frac{\delta}{\beta}$$

$$v^2{}_r = v^2{}_{\max} \sin^2 \frac{\delta}{\beta}$$

where v_t, v_r are the components of the velocity vector in the direction of the straight lines and perpendicular to it, and $\beta = \sqrt{(\alpha + 1)/(\alpha - 1)}$. Hence we have

$$\left(\frac{v_r}{v_{\max}}\right)^2 + \left(\frac{v_t}{v_{\min}}\right)^2 = 1$$

For the case, when the straight lines form a pencil of concurrent lines ("centered simple wave") the equation of the streamlines can be expressed explicitly in polar coordinates:

$$r = r_0 \cos^{-\beta^2} \frac{\delta}{\beta}$$

In this case the system of streamlines is invariant with respect to a homothetic transformation. The complementary case to this is when the system of streamlines is invariant with respect to a rigid rotation

(axially symmetric plane flow, also called "spiral flow"). This was first discussed by Taylor (21), then studied by Bateman (22) and a number of other authors, among them Courant and Friedrichs (23). They point out, that the solution can be obtained easily by superimposing in the hodograph plane the image of the purely circulatory motion and that of the purely radial flow. Thus they obtain the Legendre potential and the Legendre stream function of the Taylor-Bateman flow:

$$\Phi = k_0\Theta + k\int_{v_s}^{v}\frac{dv}{v\rho}$$

$$\psi = -k_0\int_{v_s}^{v}\frac{\rho dv}{v} + k\Theta$$

Hence

$$x = \frac{\partial\Phi}{\partial v_x} = \left(-k_0 v + \frac{kv_y}{\rho}\right)v^{-2}$$

$$y = \frac{\partial\Phi}{\partial v_y} = \left(k_0 v_x + \frac{kv}{\rho}\right)v^{-2}$$

The study of these equations shows that outside a certain circle two distinct velocity vectors correspond to every point (x,y); inside the circle no real solution exists, while on the circle itself the magnitude and direction of the velocity belonging to both branches of the solution coincide; its radial component equals the sonic speed. This circle, on which the Jacobian of the hodograph transformation vanishes, is called the "limiting line" of the flow. It may be interpreted also as the locus of the cusp points of the streamlines, each streamline having two branches, which have to be thought of as being located in two different sheets of the plane. In the case of the Taylor-Bateman flow, it can be shown that the flow along one branch is purely supersonic and is coming from infinity; it continues to be supersonic along part of the second branch and goes over without any discontinuity into subsonic flow.

Tollmien pointed out that both the Prandtl-Meyer flow and the Taylor-Bateman flow is a solution of the more general problem to find irrotational plane gas flows for which, in a general plane orthogonal coordinate system, both velocity components are functions of one coordinate only (24). For the case in which one family of coordinate lines consists of straight lines the Prandtl-Meyer flow is obviously obtained. Tollmien proved that the only other coordinate system for which a solution exists is formed by logarithmic spirals and their limiting cases; the latter yields the Taylor-Bateman flow. For logarithmic spirals proper as coordinate lines, new solutions were deduced, evaluated and represented in diagrams. Particularly the transition from super-

sonic to subsonic flow, taking place without any discontinuity, and the field properties in the neighborhood of limiting lines were studied.

Tollmien's investigation is carried out entirely in the physical plane. Most other exact theoretical solutions have been found, however, by aid of the hodograph. The linearity of the differential equations of the Legendre potential and of the Legendre streamfunction, with the velocity magnitude v and the velocity direction θ as independent variables, making superposition of flows in the hodograph plane possible, was already utilized in Chaplygin's classic work on gas jets. On this basis families of new solutions were obtained by Ringleb (24a), using for each individual term a product of a function of v and one of θ. One of the single-term solutions can be represented as a "corner flow" for an angle of deflection of 2π. It was known since Prandtl's and Meyer's work, that the "centered simple wave" solution is valid only for corners of a limited angle. The Ringleb solution is of an entirely different nature: it does effect a full deflection, but, because of the intervening of a limiting line, does not permit the streamlines to follow immediately along an infinitely thin semiinfinite plane barrier. The Ringleb flow comes as a subsonic flow from infinity, becomes supersonic, and then subsonic again, while the Prandtl-Meyer flow is supersonic throughout.

The Jacobian of the hodograph transformation of the Ringleb flow vanishes only at the limiting line, while it vanishes everywhere for a Prandtl-Meyer flow. A hodograph of this latter type is called degenerate. To a region of the physical plane a curve (or, in the trivial case of a uniform flow, a point) corresponds. In the plane the Prandtl-Meyer flow and the uniform flow are the only flows with a degenerate hodograph. In space a hodograph is said to be degenerate if a portion of the physical space is represented in the hodograph space by a portion of a surface or a curve. Giese (24b) in America and Temple (24c) in England independently had the idea of making a complete survey and geometric analysis of all irrotational gas flows which have a degenerate hodograph. Their results include, of course, all accelerationless flows, which can be expected to yield, as a limiting case, the accelerationless flows found for incompressible flow by Merlin; whether this is the case, has not yet been ascertained.

Before we leave irrotational flow problems we mention an inverse approach, which is very different from all other investigations because it does not deal, primarily, with perfect gases in the sense defined above. In his Princeton lecture, Löwner (24d) does not assume the p, ρ-relation $p/p_0 = (\rho/\rho_0)^\alpha$ to be valid. Instead, he investigates all p, ρ-relations which would, by the Bäcklund transformation, make possible the transformation of the irrotational flow equation into the Laplace equation,

the simple wave equation, and the Tricomi equation respectively. Families of such relations defined by ordinary differential equations are found, out of which suitable ones to approximate the behavior of the perfect gas (or for that matter of almost any other compressible fluid) can be selected. Moreover, Löwner points out that the transformations in question do not form a group, and hence, by combination of two or more transformations, state equations can be obtained which are not contained in the original family. Löwner suggests that an iteration process can be based hereon, which would make it possible to come as close to the desired gas equation as one wishes.

In the field of the rotational flow of a perfect gas we have first of all to mention a simple remark made by Poritsky (25), which is not only interesting in itself, but which admits also some aerodynamic applications, especially in the study of sweptback wings. Poritsky points out that, thanks to Galileo's principle of relativity of motion with respect to any uniform translation of the coordinate system, it is possible to superpose upon any plane gas flow, rotational or irrotational, a uniform flow perpendicular to its plane. A variety of spatial flows (in Berker's mentioned terminology, "pseudo-plane flows of the second kind") is originated by this exceptional possibility of direct superposition of gas flows.

In all more complex questions concerning rotational gas flows, theoretical investigation was hampered by the mathematical difficulties caused by the presence of two "state variables" (usually density and pressure, or pressure and entropy) in the equations. Munk and Prim (26) and, independently, Hicks, Wasserman, and Guenther showed that for steady flow of a perfect gas, if instead of v the reduced velocity $\mathrm{w} = \mathrm{v}/a$ (a = ultimate velocity magnitude) is used, the density can be eliminated from the general equations, and the field properties of the flow are completely defined by the dynamic equation

$$\mathrm{w} \cdot \operatorname{grad} \mathrm{w} + (1 - w^2) \operatorname{grad} \ln p^{\frac{\alpha-1}{2\alpha}} = 0$$

and the continuity equation

$$\operatorname{div} [(1 - w^2)^{\frac{1}{\alpha-1}} \mathrm{w}] = 0$$

(w = magnitude of reduced velocity w).

This "canonical" system of equations opens new possibilities for finding rotational flows of a perfect gas.

For example, Neményi and Prim suggested in a paper published only in abstract (27), that a generalization of Poritsky's superposition principle may be possible in the form

$$\mathrm{w} = \alpha \mathrm{w}_p + \mathrm{k} w_z$$

where α and w_z are scalar functions of x, y, and $\mathbf{w}_p$ is a plane steady flow of a perfect gas. This conjecture was borne out by Prim's investigation published recently (27a). By a combination of Poritsky's simple superposition principle with the "substitution principle" which is underlying the canonical equations, he proved that the above assumption yields a solution of the gas dynamic equations, provided w_z has a constant value for each streamline of the $\mathbf{w}_p$ field, and that $\alpha = \sqrt{1 - w_z^2}$.

In an investigation, which has been outlined in a preliminary publication along with other results (28) and recently discussed in detail, Prim and Neményi have extended Tollmien's above discussed inquiry to rotational gas flow, at the same time allowing also a velocity component parallel to the elements of cylindrical coordinate surfaces, having the orthogonal coordinate curves as base curves (28a). They consider however, only isometric coordinate nets. The result is that here also the coordinate net must consist of a family of logarithmic spirals or their limiting cases, if all three components of the reduced velocity field of a nonparallel flow of a perfect gas shall be expressible as functions of only one variable. A detailed investigation of the case where the polar angle in an ordinary cylindrical coordinate system is taken as the independent variable has been made by Prim and leads to a three-parameter generalization of the Prandtl-Meyer corner flow. A full report on this investigation has been presented to the Seventh International Congress of Applied Mechanics.

Another inquiry refers to properties of plane rotational flow of a perfect gas. Neményi and Prim have investigated (29) the conditions under which streamlines coincide with the lines of constant velocity magnitude or with the lines of constant vorticity (isocurls). Here again admission of vorticity does not add anything to the geometric possibilities of the flow pattern.

In a recent paper, Prim (30) discussed conditions under which the streamlines of a plane rotational flow of a perfect gas are isometric curves. Again, concentric circles and parallel straight lines are proved to be the only possible streamline patterns; these same streamline patterns are, as is well known and obvious, consistent also with irrotational gas flow. It seems that a whole number of geometric approaches lead to the same basic solutions.

Recent investigations in the theory of gas flow show that it is fruitful to consider generalized Beltrami flows ($\mathbf{w} \times \text{curl } \mathbf{w} = 0$) rather than Beltrami flows proper ($\mathbf{v} \times \text{curl } \mathbf{v} = 0$) because every flow with a constant stagnation pressure is a generalized Beltrami flow. Neményi and Prim (30a) discussed such flow fields and showed that in such a flow the stagnation enthalpy, and hence also the ultimate velocity a is con-

stant along vortex lines as well as along stream lines. The angle between the vortex lines and the streamlines is determined by the formula

$$\tan \Theta = \frac{\dfrac{|\text{grad } a|}{a}}{\dfrac{|\text{curl } \mathbf{w}|}{w}}$$

Along with this and other general properties a number of applications to helicoidal flow patterns were given.

It should be mentioned also that Giese is extending his comprehensive survey of gas flows having a degenerate hodograph to rotational flows.

In the field of nonsteady gas flow most of the investigations are restricted to problems which contain only one space variable. To study the propagation of plane waves one considers motions which in a cartesian system x, y, z are of the form $v_x = f(x,t)$, $v_y = v_z = 0$; similarly for cylindrical waves motions $v_r = f(r,t)$, $v_z = v_\phi = 0$ in a cylindrical system; and analogously for spherical waves. All three types of motion are necessarily irrotational. In all three cases a discussion in terms of the relation between the space variable and the time for an individual particle was found to be fruitful. The (x,t) plane, or the (r,t) plane corresponds to the x,y plane of a steady plane flow, the particle lines correspond to the streamlines, and the characteristics in the x, t or r, t plane correspond to the Mach lines of the steady flow. For the simple "centered" plane wave, the equation of the particle line, is obtained in a closed form:

$$\alpha = tC(1 + kt^{-\beta^{-1}})$$

where k is a parameter which distinguishes one particle line of the same solution from the others, and C characterizes a solution as a whole (23,30b).

The method of linearizing the equations of gas dynamics by a Legendre transformation was first used by Lagrange and Poisson in connection with plane wave solutions. In a recent lecture von Mises has constructed new solutions for plane waves by this method, utilizing also the analogy between the hodograph of the steady motion and the transform of the x, t plane.

A different appreach, based upon the classic work of Hadamard (30c) is followed by Bechert (30d,30e) in his studies on plane cylindrical and spherical waves. For plane waves the exact general solutions are given for the case when

$$\alpha = \frac{2m + 3}{2m + 1} \quad (m = \text{positive integer or } 0)$$

Finally, for plane waves, von Mises suggests in an unpublished paper the assumption

$$v = \alpha(t)x^2 + \beta(t)x + \gamma(t)$$

which leads to new as well as to some classic solutions of the problem.

V. Elastostatics

In elasticity, the use of inverse and semi-inverse methods goes back to the early classics of this subject. St. Venant's theory of elastic rods was mentioned in the introduction and is known to every student of elasticity. It will not be discussed here. It should be mentioned, however, that it was extended to orthogonally nonisotropic rods by Platrier.

Two of the main sources of inverse solutions are the transformation or reformulation of the fundamental equations and the study of the singularities.

As for the former, a simple idea was suggested by Lamé and Thomson, who pointed out that since any vector field having sufficient differentiability can be represented as the sum of a gradient and a curl, the displacement field should be discussed by superposition of an irrotational and a solenoidal field. Thomson discussed, in particular, irrotational displacement fields, while Voigt studied solenoidal ones. Related to these studies are some of Boussinesq's investigations (31). In recent years Doucet (32,33) has, from the same starting point, developed new methods for obtaining wide families of solutions of the equations of elasticity.

A reformulation of the three-dimensional equations of elastostatics is due to Neuber (34). He showed that any displacement field can be expressed in terms of three harmonic functions, which leads to some explicit solution overlooked before.

Also in two-dimensional elasticity, each of the various formulations bring with them characteristic exact solutions. It is sufficient to mention three approaches: the theory of Airy's stress function; the complex stress function

$$F = \frac{2}{E}(\sigma_x + \sigma_y) + 2i\omega,$$

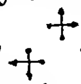
where ω is the curl of the displacement field, with its various consequences, especially the inversion method of Michell which proved so fruitful in obtaining exact solutions; and finally Carothers' reformulation (35) of the problem in terms of a pair of conjugate harmonic functions, the stress components being expressed as simple and symmetrical combinations of the "harmonic stress functions" Φ, ψ and their first derivatives:

$$\sigma_x = a\Phi + b\psi + c\left(\Phi - x\frac{\partial\Phi}{\partial x}\right) + d\left(\Phi + y\frac{\partial\Phi}{\partial y}\right)$$
$$\sigma_y = -a\Phi - b\psi + c\left(\Phi + x\frac{\partial\Phi}{\partial x}\right) + d\left(\Phi - y\frac{\partial\Phi}{\partial x}\right)$$
$$\tau_{xy} = -a\psi + b\Phi + c\left(-x\frac{\partial\Phi}{\partial y}\right) + d\left(-y\frac{\partial\Phi}{\partial x}\right)$$

In any of these presentations of the two-dimensional problem an important family of solutions emerges as amenable to particularly simple treatment. This is the family of harmonic states of stresses (Föppl's expression) characterized by $\sigma_x + \sigma_y = 0$. Föppl (36) calls them harmonic because their Airy function is a harmonic function. It is easy to see that their principal stress trajectories form an isometric net of orthogonal curves.

Starting from the presentation mentioned earlier of the displacement vector as the sum of a gradient and a curl, Thomson (1848) succeeded in giving the first solutions for a single concentrated load in an infinite solid. Related results were later given independently by Boussinesq. A decisive step in the development of stress fields with singularities is due to Love, who pointed out that a combination of two single forces, placed at a small distance, and a limiting process leads, in complete analogy to the deduction of dipoles or doublets from poles and sources in electricity and hydrodynamics, to the fields belonging to new singularities. Love calls them "typical nuclei of strain." Love and Dougall introduced the "double force without moment," the "double force with moment," the "center of dilatation," and the "center of rotation." All these are second order singularities. Among third order singularities first to be deduced was the one obtained by a limiting process from a positive and a negative center of dilatation placed at a small distance; this "double center of dilatation" was used by Love in his research on the propagation of elastic waves. In elastostatics, the first third order singularity to be used was the one which may be called a "center of bending" (obtained by a limiting process from two opposite and equal concentrated double forces with moment). This was introduced by von Kármán and Seewald (37) for the purposes of a theory of the bending of beams which are high compared to their span, and which therefore cannot be treated by aid of Navier's elementary theory of beams.

A number of new singularities were found on the basis of the following general influence principle of Neményi (38,39,40): the influence lines, influence surfaces (or in general the influence fields) of any influence in the elastic solid, can be represented by the deflection curves, deflection surfaces (or in general by the displacement field) of the same solid, if

acted upon by a singularity *dually corresponding* to the influence in question. If the influence to be studied is a strain, a stress, a stress-resultant, or a bending moment, then the dually corresponding singularity is always the limiting case of a force system in equilibrium. For example, to the dilatation ϵ_{xx} the "double force without moment" corresponds dually, while to the angular strain γ_{xy} corresponds a "center of shear," the new singularity obtained through limiting process from a scissor-like force arrangement . In the one-dimensional theory of the bending of a thin rod, there is a correspondence between the curvature of the elastic line (and hence the bending moment) and the "center of bending," while to the shear resultant of a cross section corresponds a new fourth order singularity. In the theory of bending of a plate again new third and fourth order singularities had to be introduced; for these, and for a systematic presentation of the whole subject, we refer to the papesr quoted, especially to (39), where also a broad generalization of Maxwell's principle of reciprocal displacements is given which follows immediately from the following identity, valid for any function w in which the two sets of variables $x_1, x_2 \cdots x_n$ and $\xi_1, \xi_2, \cdots \xi_n$ are interchangeable:

$$[D_{x_1,x_2\ldots\xi_1,\xi_2\ldots} w(x_1, x_2 \cdots \xi_1, \xi_2 \cdots)]_{\substack{x_1=a_1,\, x_2=a_2\cdots \\ \xi_1=\alpha_1,\, \xi_2=\alpha_2\cdots}}$$
$$= [D_{\xi_1,\xi_2\ldots x_1,x_2\ldots} w(x_1, x_2, \cdots \xi_1, \xi_2 \cdots)]_{\substack{x_1=\alpha_1,\, x_2=\alpha_2 \\ \xi_1=a_1,\, \xi_2=a_2}}$$

D being an arbitrary differential operator.

The influence field principle opened the possibility for the investigation of the stresses in elastic slabs under movable loads; the method was developed in detail by Pucher and by Dworzak (41,41a,42).

We have seen the significance of singularities up to the fourth order in the theory of thin rods and plates. It is perhaps not without interest to remark that in the one-dimensional theory of thin rods or beams, having a uniform cross section, fourth order singularities are the highest meaningful ones: singularities of a higher order than the fourth do not produce any influence outside their point of action (39).

The stress fields of some of the higher order singularities acting upon an infinite or semi-infinite elastic slice have been investigated, and the principal stress trajectories computed and drawn (43). For the bending of a plate, the curves of maximum and minimum curvature were computed which (for Poisson's modulus $m = \infty$) represent at the same time the principal trajectories of the bending moment field. It was found that the self-orthogonal system of cardioids (which is known in plane hydrodynamics as the streamline and equipotential pattern of a doublet

in a plane, cut along the semi-infinite line perpendicular to the doublet) is the principal stress trajectory system of four different types of singularities, including the double force without moment and the double center of dilatation. The same isometric system of curves is also the system of principal bending moment trajectories for four different singularities, including the center of bending (43). These and other related results made it highly probable that any isometric system of plane curves is the system of stress trajectories of a family of plane stresses which is wider than the family of "harmonic stresses," and that the isometric systems have an equally wide significance for the bending of thin plates. Indeed, Neményi proved (1933) the following theorem (44): Every isometric system of curves characterized by the analytic function $Z(z) = Z(x + iy)$ is the system of the principal stress trajectories of a five-parameter family of plane stresses, the principal stress magnitudes being given by the formulae

$$\frac{\sigma_2 - \sigma_1}{2} = (ax^2 + ay^2 + bx + cy + d)\left|\frac{dZ}{dz}\right|^2$$

$$\frac{\sigma_2 + \sigma_1}{2} = \pm R\left\{\int (2az + b + ic)\left(\frac{dZ}{dz}\right)^2 dz\right\}$$

where a, b, c, d are arbitrary real constants and R denotes the real part of the complex integral. On the basis of the well-known analogy between plate bending moments (for $m = \infty$) and the plane stress system (for any m), one can easily show that the expressions on the right represent also the quantities $\frac{m_1 - m_2}{2}$ and $\frac{m_1 + m_2}{2}$ of a system of bending moments in any portion of a plate free of loads for $m = \infty$, m_1 and m_2 being the principal bending moments.

Some interesting examples of plane stress systems characterized by isometric trajectories were given by Neuber (45). For bending of slabs, the only examples of isometric principal moment trajectories published so far are those for a center of slab bending, and for a center of slab twisting applied to an infinite slab (43).

We now give a different example. We discuss the isometric net formed by logarithmic spirals, which is characterized by the complex function $Z = e^{i\alpha} \ln z$, both as principal stress and as principal bending moment trajectories, in order to show that, while in the formulas for the infinite plane the analogy between the plane stress system and the bending of a slab is perfect, the difference in the boundary conditions, when a finite part of the plane is considered, destroys much of the analogy. The solution for the infinite plane is

$$\frac{\sigma_2 - \sigma_1}{2} = \frac{m_1 - m_2}{2} = a + b\frac{\cos\Theta}{r} + c\frac{\sin\Theta}{r} + \frac{d}{r^2},$$

$$\frac{\sigma_1 + \sigma_2}{2} = \frac{m_1 + m_2}{2}$$

$$= \pm\left[2a(\cos 2\alpha \ln r - \Theta \sin 2\alpha) - \frac{b}{r}\cos(\Theta - 2\alpha) - \frac{c}{r}\sin(\Theta - 2\alpha) + e\right]$$

Let us now cut out a concentric ring shaped region of the plane, with the limiting radii r_1 and r_2. Assuming $\alpha = 0$, $b = c = 0$ we obtain

$$\frac{\sigma_2 - \sigma_1}{2} = \frac{m_1 - m_2}{2} = a + \frac{d}{r^2}$$

$$\frac{\sigma_1 + \sigma_2}{2} = \frac{m_1 + m_2}{2} = \pm 2a \ln r + e$$

and it is obvious that by a proper choice of the three constants, we can make (either for the $+$ or for the $-$ sign) both boundaries of the ring free from loads. Thus we obtain one plane Volterra dislocation, corresponding to a translation of the two sides of a radial cut relative to each other and perpendicular to the radial direction (a generalization of this is included in Neuber's work), and one Volterra bending dislocation corresponding to a turning of the two sides of the cut relative to each other around the radius. The correspondence between plane stress and bending is here a complete one.

Now, however, let us assume $\alpha = \pi/4$, $a = b = c = e = 0$. Then we obtain $m_r = 0$, $m_t = 0$, $m_{rt} = d/r^2$. As was noticed by Kelvin and Tait, and by Kirchhoff, a shearing moment m_{rt}, if it has a constant magnitude along a boundary without any sharp corner, does not cause any deformation in a thin plate except infinitely near to this boundary. Thus, for any radius whatever, we have boundaries free from load, and hence a pure Volterra dislocation. The plate takes, as we can easily prove, the shape of an ordinary helicoidal surface; the two sides of the radial cut have a purely translatory vertical dislocation relative to each other.

On the other hand, for the plane problem $T_{rt} = d/r^2$, a harmonic state of stress implies the application, along any circular boundary, of a tangential force d/r^2 per unit length. Hence we have no Volterra dislocation here at all, but a state of stress corresponding to a definite boundary load system. It seems likely that the stress trajectories of the plane Volterra dislocation corresponding to a relative dislocation in the radial direction of the two sides of a cut are not isometric at all; they are certainly not logarithmic spirals. A somewhat more complete discussion of the family of solutions defined by $z = e^{i\alpha} \ln z$ will be given

elsewhere. Also, a new method for computing the relative deflection of any two points of a slab as a line integral of the bending moment field of the slab (representing a generalization of Mohr's method of computing beam deflection) will be discussed, since it is useful in the analysis of dislocations.

Of other inverse investigations, Suray's proof (46) that straight lines can represent one set of the principal stress trajectories of a plane stress system only if they are concentric or parallel, and Gauster's proof (47) that the magnitude of one of the principal stresses can be constant along a set of principal stress trajectories (in the absence of body forces) only if they are concentric circles or parallel straight lines, may be mentioned.

VI. Plasticity

The existing, and particularly the potential applications of inverse and semi-inverse methods in the various theories of plastic flow are numerous. Only a few of them can be mentioned here. Hencky and Prandtl were the first to call attention to the fact that in the St. Venant-von Mises theory of plasticity certain cases exist for which the equations of equilibrium together with the yield condition give the same number of equations as that of unknown stress components (48). Such problems are statically determinate, but only if the boundary conditions are appropriate. This entire theory, which proved particularly fruitful in the case of plane stress and in the case of torsion, can be considered an example of the semi-inverse approach.

We will consider only the plane statically determinate problems and discuss certain specific inverse approaches paralleling those, which were discussed above for the plane elastic system.

Using the same symbols as were used there, the Lamé equations of equilibrium of a two-dimensional continuum can be written

$$\frac{\tau}{\rho_1} = \frac{1}{2}\frac{\partial(\sigma + \tau)}{\partial s_2}$$
$$\frac{\tau}{\rho_2} = \frac{1}{2}\frac{\partial(\sigma - \tau)}{\partial s_1}$$

As to the yield conditions, it is easy to see that for the case of a *plane strain system* both the "theory of maximum shear" (Coulomb) and the theory of von Mises gives the condition τ = constant = K. Hence, whichever of these hypotheses we adopt, the Lamé equations obtain for the case of plane strain the simple form

$$\frac{\partial\sigma}{\partial s_2} = \frac{2k}{\rho_1} \qquad \frac{\partial\sigma}{\partial s_1} = \frac{2k}{\rho_2}$$

Let us ask now, under what condition such a net of trajectories is isometric.

Isometry of a net makes it possible to identify it with the net of streamlines and equipotential lines of a plane irrotational flow field having a complex flow potential $Z = \phi + i\psi$. If the velocity magnitude of this flow field is denoted by v, obviously

$$\frac{\partial \ln v}{\partial s_2} = \frac{1}{\rho_1} \qquad -\frac{\partial \ln v}{\partial s_1} = \frac{1}{\rho_2}$$

Hence ρ_1 and ρ_2 can be eliminated from the equations of equilibrium and we have

$$K \frac{\partial \ln v}{\partial s_2} = \frac{1}{2} \frac{\partial \sigma}{\partial s_2}$$
$$K \frac{\partial \ln v}{\partial s_1} = -\frac{1}{2} \frac{\partial \sigma}{\partial s_2}$$

and hence

$$\ln v = g(s_2) - f(s_1) = G(\psi) - F(\phi)$$

From the fact that this $\ln v$ is harmonic function of ϕ and ψ it follows immediately that it must have the form $a(\phi^2 - \psi^2)$ plus a linear function of ϕ and ψ. Hence we can see that the equation $z = \int e^{aZ^2+(b+ic)Z+(d+if)} dZ$ defines the only isometric nets which satisfy the conditions of plane plastic strain. As we know from Boussinesq's work that the main trajectories of plane plastic strain form under suitable parametrization, an iso-areal net, we see that the only iso-areal net which, at the same time, is isometric is given by the above deduced mapping function. Essentially the same result was obtained by Carathéodory and Schmidt by a different deduction, which started from the lines of maximum shear stress (an orthogonal net of curves which intersect the principal stress trajectories at 45° angles).

From the same equations of equilibrium we also can see immediately that one of the principal stresses can have a constant magnitude along each trajectory belonging to one set only if they are concentric circles or parallel straight lines.

From the study of the lines of maximum sheer stress von Mises deduced the following exact solution of the plane plastic strain problem

$$z = \sum_{\nu} \frac{k_\nu c_\nu}{i - c_\nu} e^{c_\nu \alpha + \frac{\beta}{c_\nu}}$$

where the k_ν are real, and the c arbitrary real or imaginary constants; α, β are the parameters of the two maximum shear line systems. He showed

a graphic method for the superposition of terms of this type. In this inverted form the problem is a linear one.

In the case of plane stress the hypothesis of Coulomb still leads to the yield condition $\tau = K$ but the hypothesis of von Mises gives the nonlinear yield condition

$$\tau = \frac{1}{\sqrt{3}} \sqrt{K^2 - \sigma^2}$$

The general theory, and many exact solutions, have been worked out quite recently by Geiringer, and presented to the second Plasticity Symposium of Brown University.

In his lecture at the first Plasticity Symposium at Brown University in 1948, von Mises pointed out that the mentioned yield condition can be approximated adequately by a much simpler one of the form

$$\tau = \pm a^2\sigma^2 \mp \frac{1}{4a^2} = \pm \left(a\sigma + \frac{1}{2a}\right)\left(a\sigma - \frac{1}{2a}\right)$$

Some implications and applications of this yield condition have also been pointed out by Geiringer in her mentioned lecture.

We will give a different application of this yield equation: We will find the isometric principal trajectory systems consistent with it, thus giving a parallel solution to that of Carathéodory and Schmidt.[3]

Replacing the curvatures in the Lamé equations by the appropriate expressions involving the velocity magnitude v we have

$$\tau \frac{\partial \ln v}{\partial s_2} = \frac{1}{2} \frac{\partial}{\partial s_2} (\sigma + \tau) \qquad \tau \frac{\partial \ln v}{\sigma s_1} = \frac{1}{2} \frac{\partial}{\partial s_1} (\tau - \sigma)$$

Introducing the simplified yield condition into this system of equations the latter can be brought into the simple form:

$$\frac{\partial \ln v}{\partial s_2} = \frac{\partial \ln \left(\sigma - \frac{1}{2a^2}\right)}{\partial s_2}$$

$$\frac{\partial \ln v}{\partial s_1} = \frac{\partial \ln \left(\sigma + \frac{1}{2a^2}\right)}{\partial s_1}$$

Hence

$$\ln v = \ln \left(\sigma - \frac{1}{2a^2}\right) + \ln F_1(\phi) = \ln \left(\sigma + \frac{1}{2a^2}\right) + \ln G, (\psi)$$

[3] The new solution given here is due to joint work of Andrew Van Tuyl and the author.

and therefore

$$\frac{v}{F_1(\phi)} + \frac{1}{2a^2} = \frac{v}{G_1(\psi)} - \frac{1}{2a^2}$$

$$v\left[\frac{1}{F_1(\phi)} - \frac{1}{G_1(\psi)}\right] = -\frac{1}{a^2}$$

Hence

$$\ln v = -\ln\left\{a^2[G(\psi) + F(\phi)]\right\}$$

We have to find therefore a harmonic function of ϕ and ψ, which has the form

$$\ln\,[G(\psi) + F(\phi)]$$

F and G have to satisfy then the ordinary differential condition

$$FF'' - F'^2 + GG'' - G'^2 + GF'' + FG'' = 0$$

It follows that

$$G(\psi)F''(\phi) + F(\phi)G''(\psi) = M(\phi) + N(\psi)$$

In order to find the most general functions F and G which satisfy this relation we differentiate first with respect to ϕ than to ψ and obtain

$$G'(\psi)F'''(\phi) + F'(\phi)G'''(\psi) = 0$$

There are two distinct ways to satisfy this equation trivially. Either we can put quadratic functions both for G and F, in which case the original differential condition restricts them to $G = (\psi + \beta)^2$, $F = (\phi + \alpha)^2$. This leads to the complex function

$$Z = C(z + A)^{\frac{1}{2}} + B \quad (A,\ B \text{ and } C \text{ arbitrary complex numbers})$$

Alternatively, we can choose either of the two functions F and G identically 0; by virtue of the original differential equation we obtain then $F = Ce^{c\phi}$ (C and c real numbers). This leads to concentric circles and radial straight lines as principal stress trajectories.

Finally, if neither of the two terms vanishes we have the simultaneous differential equations

$$\frac{F'''}{F'} = -\frac{G'''}{G'} = c^2$$

From this we obtain

$$F(\phi) + G(\psi) = \cosh\,(c\phi + B_1) + \cos\,(c\psi + B_2)$$

hence
$$\frac{dZ}{dz} = \cosh^{-2}\left(\frac{cZ + B_1 + iB_2}{2}\right)$$

and we obtain the family of stress trajectories defined by the mapping function

$$A + Z + \frac{1}{c} \sinh (cZ + B) = z$$

(A, B complex, c real constant).

A comparison of the new solution with that of Carathéodory and Schmidt shows that the change of the yield condition changes the solution completely: only the trivial case of concentric circles and radial straight lines is common to both solutions.

For other recent inverse solutions for various models of plastic materials we refer to the books of Prager, Sokolovsky, and Van Iterson (48a,48b,48c).

VII. Significance and Heuristic Value of the Inverse and Semi-Inverse Methods of Mechanics

1. Inverse and semi-inverse methods lead to solutions of important boundary value problems. We have seen many examples of this. Often out of the family of semi-inverse solutions, the solution for important boundary value problems can be directly chosen, as was done, for example, by Neuber in the case of the elastic, plane, ring-shaped region with non-concentric boundaries. In other cases, synthesis of the basic solutions leads to approximate solutions of important boundary value problems. In hydrodynamics, the classic example of this is, of course, Rankine's source sink method, for irrotational nonviscous flow from which the modern singularity methods of hydrodynamics have followed. These resulted in a reduction of flow problems to integral equation and to graphic methods for the solution of the latter.

In theory of elasticity the synthesis of solutions belonging to the singularities has not yet been developed to a similar degree of versatility. Although Betti had already discussed the possibility of a synthesis (31) its application to the solution of difficult boundary value problems was first attempted by Miche (49). His work was followed up by Weinel and others.

We have seen, in connection with Löwner's work, that in certain cases an iterative process can lead from an inverse solution to an arbitrarily close approximation to an exact solution of a definite problem.

2. Inverse and semi-inverse methods may lead to the discovery of unsuspected discontinuities, limitations, or general field properties of the solution of certain differential equations. In these purely heuristic and suggestive possibilities lay probably the greatest value of special solutions.

A major example of this is Tollmien's study of the limiting lines of plane irrotational gas flow in the case of flow fields expressible in terms of one curvilinear coordinate. Four years later he succeeded in giving a general theory of the limiting lines which are interesting from the physical as well as from the geometric standpoint (50).

While each new formulation of the general equations of a problem brings with it a number of basic, often new, solutions, it may also happen that inverse solutions suggest a reformulation of the general theory. Thus, for example, Wegner has generalized Neményi's theorem on isometric principal stress trajectories which was discussed above. Wegner showed that any plane elastic stress field, in the absence of body forces, can be obtained through superposition of two suitably chosen stress fields, both having isometric trajectory patterns. One of them may be chosen as a harmonic state of stress.

The theorem on isometric trajectory patterns may have also motivated Prager's (51) formal analogy of the formulation of the plane stress problem in terms of the conjugate stress deviation to the formulation of the problem of irrotational incompressible flow in terms of the conjugate velocity function.

3. Inverse investigations may settle existence questions in a positive sense, or may decide a uniqueness problem in a negative sense. Also they suggest new uniqueness questions, and conjectures for their answers. For example, we may ask whether, similar to the special case of isometric stress trajectory patterns, any given plane orthogonal stress trajectory pattern of an elastic plate is consistent with a five-parameter family of stress distributions.

A distinct advantage of inverse solutions, if they are expressed in terms of familiar functions, is that they make possible limiting processes. The difficult, and so far unsolved question of the extent to which a nonviscous incompressible flow can be considered a limiting case of the flow of a perfect gas may be approached with the aid of inverse solutions. Many other simplifications or idealizations may be examined in the light of specific solutions obtained by the inverse method.

Of course a positive result of an examination of a hypothesis in light of a specific example is never decisive. A negative result may be much more important. For example, the fact that the "Chaplygin gas" ($\alpha = -1$) behaves, as Prim's investigation has proved, in the case of its rotational flow past a corner, fundamentally different from any familiar gas ($1 < \alpha < 1\frac{2}{3}$) is a decisive reason to doubt the physical validity of any investigations based on that assumption.

4. Semi-inverse methods are essential for the comparative study of the differential equations of the various problems of mechanics.

	Fluid Dynamics			Elasticity		Plasticity	
	Nonviscous, incompressible fluid (ideal fluid)	Viscous, incompressible fluid	Perfect gas	Plate stressed in its own plane	Thin plate bent (and twisted)	Plastic plate under plane strain Coulomb's yield condition	Plastic plate under plane stress von Mises parabolic yield condition
Streamlines coincide with isovels	Streamlines concentric circles or parallel straight lines	Answer seems to be unknown	Streamlines concentric circles or parallel straight lines (Neményi and Prim)	Constant principal stress along each trajectory of a system of main trajectories possible only if they are concentric circles or parallel straight lines. (Gauster)	Constant principal curvature along each trajectory of a system of main trajectories possible only if they are concentric circles or parallel straight lines	Constant principal stress along each trajectory of a system of main trajectories possible only if they are concentric circles or parallel straight lines	Constant principal stress along each trajectory of a system of main trajectories is possible only if they are concentric circles or parallel straight lines
Isocurls coincide with streamlines	Any rotational plane flow has this property (Helmholtz)	(1) Any flow which has a constant curl throughout the field. (2) Streamlines concentric circles or parallel straight lines (Kampé de Fériet)	Streamlines concentric circles or parallel straight lines (Neményi and Prim)				
Isocurls coincide with isovels	Streamlines concentric circles or parallel straight lines	Answer seems to be unknown	Streamlines concentric circles or parallel straight lines (Neményi and Prim)				
Streamlines and their orthogonal trajectories form isometric net	(1) Any irrotational flow (2) Rotational flow only if streamlines concentric circles or parallel straight lines (Prim)	(1) Any irrotational flow (2) Rotational flow only if streamlines logarithmic spirals or their limiting cases (concentric circles, radial straight lines, parallel straight lines) (Jeffery, Hamel)	(1) Irrotational flow: streamline concentric circles parallel straight lines, or radial straight lines (Gilbarg) (2) Rotational flow: streamlines, concentric circles, or parallel straight lines (Prim)	Any isometric net of plane curves can be interpreted as the system of principal stress trajectories for a five-parameter family of stresses (Neményi)	Any isometric net of plane curves can be interpreted as the system of trajectories of principal curvature for a five parameter family of states of deformation (Neményi)	The only isometric nets of principal stress trajectories are defined by the mapping function $z = \int e^{aZ^2+bZ+c} dZ$ B, C complex, a real constant (Carathéodory-Schmidt)	The only isometric nets of principal stress trajections are defined by the mapping functions: $Z = C(z + A)^{\frac{1}{2}} + B$ $Z = C_1 - \frac{1}{a}\log(C_2 - z)$ $z = A + Z + \frac{1}{c}\sinh(cZ + B)$ (A, B, C, C_1, C_2 complex, a, c real) (Van Tuyl-Neményi)
Isocurls and their orthogonal trajectories form isometric net	Streamlines concentric circles or parallel straight lines	Streamlines concentric circles or parallel straight lines. (Jeffery)	Answer seems to be unknown				

By putting the same geometric or kinematic restrictions upon the solution of various differential equations and comparing the results, new insight into their common properties and their distinguishing features can be gained. The analysis of the results of such a comparison involves, as Garrett Birkhoff points out, group theoretical methods.

The answers for the steady flows of three different idealized fluids to five interrelated kinematic-geometric questions are given in the table attached to this paper. Answers to two analogous questions of two-dimensional elasticity and plasticity have been added to the table. All these results have been proved only for the case of the absence of an external body force field.

Acknowledgement

The author is indebted to C. A. Truesdell and Andrew Van Tuyl for very valuable suggestions, and to R. C. Prim for constructive criticism.

References

1. Rankine, W. J. M., On plane waterlines in two dimensions, Philosophical Transactions 1864. See also Miscellaneous Scientific Papers, London, 1881.
2. Hamel, G., Potentialströmungen mit konstanter Geschwindigkeit, *Sitzber. preuss. Akad. Wiss. Physik. math. Klasse*, 5–20, 1937.
3. Castoldi, L., Sopra una proprietà dei moti permanenti di fluidi incompressibili in cui le linee di correnti formano una congruenza normale di linee isotache, *Atti accad. naz. Lincei, Classe sci. fis., mat. e nat.*, [8], **3**, 333–337 (1947).

3a. Prim, R. C. Steady rotational flow of ideal gases. Dissertation. Princeton, 1949.

4. Merlin, E., Étude du mouvement d'un fluide parfait dépourvu d'accélération, *Ann. école normale supérieur*, [3], **55**, 223–255 (1938).
5. Gilbarg, D., On the flow patterns common to certain classes of plane fluid motions, *J. Math. Phy.* **26**, 137–142 (1947).
6. Prim, R. C., On the uniqueness of flows with given streamlines, *J. Math. Phys.*, XXVIII, 1, 50–53 (1949).
7. Beltrami, E., Considerazioni idrodinamiche, *Rend. ist. Lombardo sci.*, **22**, 121–130 (1889).
8. Massotti, A., Moti rotazionali a strati piani di un liquido perfetto, *Rend. ist. lombardo sci.*, [2], **59**, 82–86 (1926).
9. Nöther, F., Integrationsprobleme der Navier-Stokesschen Differentialgleichungen, Handbuch der physikalischen und technischen Mechanik, **5**, pp. 719–796, Leipzig, 1930.
10. Rosenblatt, A., Solutions exactes des équations du mouvement des liquides visqueux, Memorial des sciences mathématiques, Fasc. 72, Paris, 1935.
11. Berker, Ratib, Sur quelques cas d'intégration des équations du mouvement d'un fluide visqueux incompressible, Paris-Lille, 1936.
12. Jeffery, G. B., On the equations of motion of a viscous fluid, Phil. Mag., **29**, 445–454 (1915).
13. ——— On the two-dimensional steady motion of a viscous fluid, Phil. Mag., **29**, 455–464 (1915).

14. HAMEL, G., Spiralförmige Bewegungen zäher Flüssigkeiten, Jahresberichte der deutschen Mathematiker Vereinigung 25, pp. 34–60, 1916.
15. OSEEN, C. W., Exakte Lösungen der hydrodynamischen Differentialgleichungen, Opuscula Mathematica A. Wiman dedicata, pp. 147–156, Lund, 1930.
15a. GÖRTLER, H., AND WIEGHARDT, K., Über eine gewisse Klasse von Strömungen zäher Flüssigkeiten und eine Kennzeichnung der Poiseuille Strömung. *Math. Z.*, 48, 247–250 (1942).
16. BATEMAN, HARRY, Report of the Committee on Hydrodynamics, *Bull. Natl. Research Council (U.S.)*, No. 84, 634 (1932).
17. TAYLOR, G. I., On the decay of vortices in a viscous fluid, *Phil. Mag.*, 46, 671–674 (1923).
18. KAMPÉ DE FÉRIET, J., Détermination des mouvements plans d'un fluide visqueux incompressible où le tourbillon est constant le long des lignes de courant, Comptes rendus du 9*e* Congrès International des Mathématiciens, 2, pp. 298–299, Zürich, 1932.
19. TRKAL, V., Poznamka k hydrodynamice vazkych tekutin, *Časopis pro Pestovani Matematiky a Fisiky* 48, 302–311 (Praha 1919).
19a. BALLABH, R., Superposable Fluid Motions. *Proc. Benares Math. Soc.*, 2, 69–79 (1940).
19b. ——— Superposable Motions in Heterogeneous Incompressible Fluids, *Proc. Benares Math. Soc.*, 3, 1–9 (1941).
19c. DOUCET, E., Études de quelques solutions des équations de l'hydrodynamique. *Assoc. franç. avance. sci., Sci. Sections* 63, 41–58 (1941).
20. MEYER, TH., Über zweidimensionale Bewegungsvorgänge in einem Gas das mit Überschallgeschwindigkeit strömt. (Dissertation Göttingen), Forschungsarbeiten auf dem Gebiete des Ingenieurwesens, 62, Berlin, 1908.
21. TAYLOR, G. I., Recent work on the flow of compressible fluids, *J. London Math. Soc.*, 5, 234 (1930).
22. BATEMAN, H., Irrotational motion of a compressible inviscid fluid, *Proc., Natl. Acad. Sci.*, 16, 816–825 (1930).
23. COURANT, R., AND FRIEDRICHS, K. O., Supersonic flow and shock waves, New York, 1948.
24. TOLLMIEN, W., Zum Übergang von Unterschall in Überschall Strömungen, *Z. angew. Math. Mech.*, 17, 117–136 (1937).
24a. RINGLEB, F., Exakte Lösungen der Differentialgleichungen einer adiabatischen Gasströmung, *Z. angew. Math. Mech.*, 20, 185–198 (1940).
24b. GIESE, J. H., Compressible fields with degenerate hodographs, Abstracts, *Bull. Am. Phys. Soc.*, Program of the New York meeting, January 30–31, 1947.
24c. TEMPLE, G. F. J., Symposium Lecture on recent developments in compressible flow, New York meeting. *Am. Phys. Soc.*, January 30–31, 1947.
24d. LÖWNER, CHARLES, Lecture on the theory of irrotational gas flow, Princeton University, March 1948.
25. PORITSKY, H., Compressible flows obtainable from two-dimensional flows through addition of a constant normal velocity, *J. Applied Mechanics*, 13, A61–65 (1946).
26. MUNK, M. M., AND PRIM, R. C., On the canonical form of the equations of steady motion of a perfect gas, *J. Applied Phys.*, 19, 10,957–8 (1948).
27. NEMÉNYI, P. F., AND PRIM, R. C., On methods of deriving three-dimensional gas flow patterns from plane flows, Abstract. Bull. *Am. Phys. Soc.*, Program of the 1947 Annual Meeting, New York City, January 29–31, p. 24, 1948.
27a. PRIM, R. E., A new superposition principle for steady gas flows. Quarterly of Applied Mathematics, 7, 445–450 (1950).

28. NEMÉNYI, P. F. AND PRIM, R. C., Some properties of the rotational flow of a perfect gas, *Proc. Natl. Acad. Sci.*, March 1948.
28a. PRIM, R. C., AND NEMÉNYI, P. F. On a family of rotational spatial gas flows. Quarterly Journal of Mech. and Applied Math., **2**, 129–135 (1949).
29. ——, Some properties of plane rotational gas flow, *J. Math. Phys.*, **27**, 130–135 (1948).
30. PRIM, R. C., On the existence of steady gas flow in plane isothermal streamline patterns, *Bull. Am. Math. Soc.*, **54**, No. 10 (October 1948).
30a. NEMÉNYI, P. F., AND PRIM, R. C., On the Steady Beltrami Flow of a Perfect Gas. Proceedings of the 7th International Congress of Applied Mechanics, Vol. 2, pp. 300–314, London, 1948.
30b. VON MISES, R., Lectures on the Mathematical Theory of compressible fluid flow. (Mimeographed), Cambridge, Mass., 1948.
30c. HADAMARD, J., Leçons sur la propagation des ondes, Paris, 1900.
30d. BECHERT, K., Zur Theorie Ebener Störungen, reibungsfreier Gase, *Ann. Physik*, **37**, 89–123 (1940).
30e. ——, Über die Ausbreitung von Zylinder und Kugelwellen in Reibungsfreien Gasen und Flüssigkeiten, *Ann. Physik*, **39**, 169–202 (1941).
31. For all classical investigations referred to in this part of the paper see A. E. H. LOVE, *A Treatise on the Mathematical Theory of Elasticity*, 4th Edition New York, 1944, and TEDONE AND TIMPE, *Spezielle Ausführungen zur Statik elastischer Körper.* Enzyklopädie der Mathematischen Wissenschaften IV, **4**, pp. 125–214, Leipzig, 1907–1914.
32. DOUCET, E., Intégration des équation réduites de l'équilibre élastique, Proceedings of the 5th International Congress of Applied Mechanics, pp. 255–257, Cambridge, Mass., 1938.
33. ——, Prémière étude de la théorie de l'élasticité, Paris, 1937.
34. NEUBER, H., Ein neuer Ansatz zur Lösung räumlicher Probleme der Elastizitätstheorie, *Z. angew. Math. Mech.*, **14**, 203–212 (1934).
35. CAROTHERS, S. D., The direct determination of stresses in two dimensional elastic systems, *London Roy. Soc. Proc.*, **A97**, 110 (1920).
36. FÖPPL, L., Konforme Abbildung ebener Spannungszustände, *Z. Angew. Math. Mech.*, **11**, 81–92 (1931).
37. VON KÁRMÁN, TH., Über die Grundlagen der Balkentheorie, pp. 3–11, SEEWALD, F., Die Spannungen und Formänderungen von Balken mit rechteckigem Querschnitt, Abhandlungen aus dem Aerodynamischen Inst. Aachen, pp. 11–34, 1927.
38. NEMÉNYI, P. F., Über eine neue Singularitätenmethode für die Elastizitätstheorie, *Z. angew. Math. Mech.*, **9**, 488–490 (1929).
39. ——, Über die Singularitäten der Elastizitätstheorie *Z. Angew. Math. Mech.*, **10**, 383–399 (1930).
40. ——, Selbstspannungen elastischer Gebilde, Handbuch der physikalischen und technischen Mechanik, **4/2**, 199–211 (1931).
41. PUCHER, A., Die Momenteneinflussfelder rechteckiger Platten, Deutscher Ausschuss für Eisenbetonbau, Heft 90, Berlin, 1938.
41a. ——, Über die Singularitätenmethode an elastischen Platten, *Ingenieur archiv*, **12**, 76–100 (1941).
42. DWORZAK, W., Der freie Rand an rechteckigen Platten, *Osterr. Ingenieur archiv*, **1**, 66–77 (1947).
43. NEMÉNYI, P. F., Über Spannungsfelder die mit bekannten Strömungsfeldern isomorph sind, Proceedings, 3rd International Congress of Applied Mechanics, **2**, pp. 155–167, Stockholm, 1931.

44. ———, Stromlinien und Hauptspannungstrajektorien, *Z. angew. Math. Mech.*, **13**, pp. 364–366 (1933).
45. Neuber, H., Der ebene Stromlinienspannungszustand mit lastfreiem Rande, *Ingenieur archiv*, **6**, 325–334 (1935).
46. Suray, Saffet, Sur les lignes de tension principale dont une famille est constituée des lignes droites, *Rev. fac. sci. univ. Istanbul*, **A6**, 83–87 (1941).
47. Gauster, W., Bemerkungen zum ebenen Spannungszustand, *Z. angew. Math. Mech.*, **5**, 519–521 (1925).
48. For all publications prior to 1937 referred to in this part of the paper, see: H. Geiringer and W. Prager, Mechanik isotroper Körper im plastischen Zustande, *Ergeb. exacten Naturw.*, **13**, Berlin, 1934; and Hilda Geiringer, Fondaments mathématiques de la théorie des corps plastiques isotropes. Fascicule 86. Memorial des Sciences Mathématiques, Paris, 1937 (89 pages).
48a. Prager, W., Theory of Plasticity (Mimeograph), Providence, 1942.
48b. Sokolovsky, V. V., Theoria Plastinosty (Theory of Plasticity), Moscow-Leningrad, 1946 (306 pages).
48c. Van Iterson, Plasticity in Engineering, Glasgow, 1948.
49. Miche, R., Le calcul pratique des problèmes élastiques à deux dimensions par la méthode des équation intégrales. Proceedings, 2nd International Congress of Applied Mechanics, pp. 126–130, Zurich, 1927.
50. Tollmien, W., Grenzlinien adiabatischer Potentialströmungen, *Z. angew. Math. Mech.*, **21**, 140–161 (1941).
51. Prager, W., Streamlines and lines of principal stress, *Rev. Math. Union Interbalcanique*, **3**, 63–65 (1941).

Theory of Filtration of Liquids in Porous Media*

By P. Ya. Polubarinova-Kochina, *Corresponding Member of the Academy of Sciences of the U.S.S.R., Moscow, U.S.S.R.*, and S. B. Falkovich, *Moscow, U.S.S.R.*

CONTENTS

I. Fundamental Equations of Steady Motion ... 154
1. Darcy's Law ... 154
2. The Problem of Soil-Water Motion Which Does Not Obey Darcy's Law ... 155
II. Motion of Soil Waters without a Free Surface ... 157
1. Impermeable Walls ... 157
2. Boundaries of a Water Reservoir ... 157
3. Problem of the Flow around an Oblique Cutoff ... 159
4. Filtration under the Apron ... 161
III. Motions with a Free Surface—Zhukovsky's Function ... 163
1. Zhukovsky's Problem—the Flow around Cutoffs ... 163
IV. Motion with a Free Surface. Velocity Hodograph ... 165
1. Filtration from a Channel of a Trapezoidal Cross Section ... 166
2. Motion of Soil Water through a Dam Drained at the Base ... 168
V. The Seepage Line ... 171
1. A Triangular Earth Dam Filled with Water ... 172
2. Drain Channel of Trapezoidal Cross Section ... 173
VI. General Case of the Velocity Hodograph ... 175
VII. Application of the Analytic Theory of Differential Equations ... 176
1. Seepage of Soil Water onto the Downstream Dam Slope ... 180
VIII. Inverse Methods ... 182
1. Filtration from Channels ... 183
2. Inflow from Infinity to the Drain Channel ... 183
3. Earth Dam with a Curvilinear Boundary of the Reservoir ... 183
4. Dam with a Curvilinear Upstream Slope, but with a Horizontal Bottom of the Upper Reservoir ... 185
5. Dams with a Nonhorizontal Impermeable Rock ... 186
IX. Other Methods ... 187
1. Reduction to Hilbert's Problem ... 187
2. Applications of Functional Analysis ... 191
3. Variational Methods ... 192
X. Filtration in Multilayer Soils ... 193
1. Construction of the Flow from Singularities ... 193
2. Construction of the Flow by Means of the Fourier Integral ... 195
3. Three-Dimensional Sink in a Two-Layer Medium ... 196
4. Method of Differential Equations ... 197
5. Hydraulic Theory of a Well in a Two-Layer Medium ... 199

* Translated by D. R. Mazkevich, Executive Editor, *Applied Mechanics Reviews*. Article appeared originally in *PMM*, XI (1947).

XI. Filtration of Oil 201
1. Inflow of Liquid to Wells in a Heterogeneous Medium 205
XII. Three-Dimensional Problems 206
XIII. Unsteady Flows 210
1. Flow under Dams at Variable Pressures 210
2. The Problem of Contraction of an Oil-Bearing Region 211
3. Problem of the Depression of the Soil-Water Level under the Action of a Drain Pipe 213
4. Problem of the Displacement of a Liquid Surface 215
5. The Motion of Two Liquids with Different Densities and Viscosities 216
6. Elastic Regime 217
References 217

The aim of this paper is to present the development and the current state in the Soviet Union of the theory of filtration of liquids, mainly of the part of it which deals with exact solutions. We will not touch upon the theory of filtration of gases and mixtures of gas and liquid, where almost no exact solutions are known.

I. Fundamental Equations of Steady Motion

1. Darcy's Law

Owing to Darcy's experiments, consisting in forcing water through sand in a straight tube, and to contemporary experiments, using a model of a petroleum layer with voids, one can regard as established that in fine-grained soils and for the usual velocities, the following Darcy's law holds: The steady flow of liquids in a porous medium is sufficiently well described by the equations

$$v = \operatorname{grad} \varphi = -\kappa \operatorname{grad} H, \quad H = p/(\rho g) + y \tag{1.1}$$

where v is the vector of filtration velocity, κ the filtration coefficient, H the pressure function, p the pressure, ρ the density, g the gravitational acceleration, and y the vertical coordinate (positive upward).

The filtration velocity is the volume discharge per unit time and unit area, so that, for instance, for one-dimensional motion

$$v_x = \frac{mdx}{dt} \tag{1.2}$$

where dx/dt is the velocity of a liquid particle and m the porosity of the ground, i.e., the ratio of the voids area to the total area occupied by the voids and the ground.

The continuity equation for an incompressible fluid has the usual form

$$\operatorname{div} v = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} = 0 \tag{1.3}$$

where v_x, v_y, v_z are the projections of the filtration velocity on the coordinate axes.

The velocity potential

$$\varphi = -\kappa\left(\frac{p}{\rho g} + u\right) + C \tag{1.4}$$

satisfies Laplace's equation on account of equation (1.3).

For plane motion there exists a complex potential

$$f = \varphi + i\psi, \qquad \frac{df}{dz} = v_x - iv_y \tag{1.5}$$

In Russia the first work on water motion in sands was done by Zhukovsky (1889) who chiefly studied the problem of the flow of liquid into oil wells.

2. The Problem of Soil-Water Motion Which Does Not Obey Darcy's Law

The problem is analyzed in Khristianovich's (65) paper. With the help of the notion of the hydraulic gradient $J = -dH/ds$, one can express the filtration law in the form

$$J = \phi(v), \quad (v^2 = v_x^2 + v_y^2 + v_z^2)$$

For Darcy's law $J = v^\gamma/\kappa$. In some cases experimental data lead to relations of the form

$$J = v^\gamma/\kappa, \quad J = av + bv^2 + cv^3, \text{ etc.} \tag{1.6}$$

Khristianovich considers equations of the form (v is the vector of filtration velocity)

$$J = -\text{grad } H = \frac{v\phi(v)}{v} = \frac{v}{K}, \quad \left(K = \frac{v}{\phi(v)}\right) \tag{1.7}$$

For plane motion, eliminating H from these equations, and adjoining the continuity equation of the incompressible fluid, we get the system

$$\frac{\partial(v_x/K)}{\partial y} - \frac{\partial(v_y/K)}{\partial x} = 0, \quad \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} = 0 \tag{1.8}$$

Introducing the velocity vector v and its angle θ with the x axis, instead of v_x, v_y, and taking H and ψ as the independent variables, the flow function ψ will satisfy the equations

$$\frac{\partial\psi}{\partial x} = -v_y = -v \sin\theta, \qquad \frac{\partial\psi}{\partial y} = v_x = v\cos\theta$$

and we get

$$\frac{\partial \theta}{\partial \psi} - \frac{\phi(v)}{v^2}\frac{\partial v}{\partial H} = 0, \qquad \frac{\partial \theta}{\partial H} + \frac{v\phi'(v)}{\phi(v)}\frac{\partial v}{\partial \psi} = 0 \tag{1.9}$$

Further, the author reduces this system to a system of four equations, in which the functions v, θ, ψ, and H are expressed in terms of the auxiliary independent variables μ and ν:

$$\frac{\partial \psi}{\partial \nu} + \frac{v}{\phi(v)}\sqrt{\frac{v\phi'(v)}{\phi(v)}}\frac{\partial H}{\partial \mu} = 0, \qquad \frac{\partial \psi}{\partial \mu} - \frac{v}{\phi(v)}\sqrt{\frac{v\phi'(v)}{\partial(v)}}\frac{\partial H}{\partial \nu} = 0$$
$$\sqrt{\frac{\phi'(v)}{v\phi(v)}}\frac{\partial v}{\partial \nu} - \frac{\partial \theta}{\partial \mu} = 0, \qquad \sqrt{\frac{\phi'(v)}{v\phi(v)}}\frac{\partial v}{\partial \mu} + \frac{\partial \theta}{\partial \nu} = 0 \tag{1.10}$$

Finally, a fictitious filtration velocity is introduced

$$s = \log \bar{v} = \int \sqrt{\frac{\phi'(v)}{v\phi(v)}}\, dv \tag{1.11}$$

and then the system (1.10) is replaced by

$$\frac{\partial \log \bar{v}}{\partial \nu} - \frac{\partial \theta}{\partial \mu} = 0, \qquad \frac{\partial \log \bar{v}}{\partial \mu} + \frac{\partial \theta}{\partial \nu} = 0$$
$$\frac{\partial H}{\partial \mu} = -L\frac{\partial \psi}{\partial \nu}, \qquad \frac{\partial H}{\partial \nu} = L\frac{\partial \psi}{\partial \mu}, \qquad \left(L = \frac{\phi(v)}{v}\sqrt{\frac{\phi(v)}{v\phi(v)}}\right) \tag{1.12}$$

Here $\mu + i\nu = f(x + iy)$, where f is an analytic function. For Darcy's law $\bar{v} = v$, $L = 1/\kappa$. If $J = v^\gamma/\kappa$,

$$\bar{v} = v^{\sqrt{\gamma}}, \qquad L = \frac{1}{\kappa\sqrt{\gamma}}v^{\gamma-1}$$

One can interpret L as a quantity inverse of a certain fictitious filtration coefficient $L = \kappa_{\text{fic}}^{-1}$.

If the solution of the system (1.12) is found, one can find x,y from the equations

$$x = -\int \frac{\cos\theta}{\phi(v)}\, dH + \frac{\sin\theta}{v}\, d\psi, \qquad y = -\int \frac{\sin\theta}{\phi(v)}\, dH - \frac{\cos\theta}{v}\, d\psi$$

Khristianovich proposes to solve the system (1.12) in the following manner. Assume in the plane $\mu\nu$ a region analogous to the region given in the plane xy or simply coinciding with it, and solve the first two equations (1.12); then substitute the value of v thus obtained into L, getting thus $L(\mu,\nu)$, and solve the second equations (1.12). Thereby the conditions on the rigid walls and on the boundaries of the wells and the soil

will be the same as in the case of Darcy's law (see section II); on the free surface and on the seepage interval they will be, however, different (cf. sections III and IV).

The paper by Khristianovich (65) offers a method for an approximate solution of the problem. A special section is devoted to the application of the electric analogy method to the study of a motion which does not obey Darcy's law.

II. Motion of Soil Waters without a Free Surface

The systematic development of the filtration theory in the U.S.S.R. began in 1922 with Pavlovsky's paper (113). In all rigor (including the uniqueness theorem), it studies the problem of plane motion of soil waters under dams, under the action of the difference of the heads H_1 and H_2 in the upper and lower water, the region of motion being composed of rectilinear segments.

The boundaries of the filtration region can be of two types.

1. Impermeable Walls

The lower base of the dam; cut offs $l_1, l_2, \cdots l_p$, (Fig. 1), i.e., sections of rows of piles driven into the soil; the boundary between a permeable

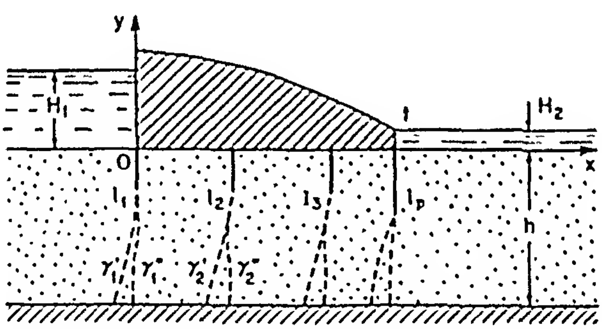


Fig. 1

and impermeable rock. These boundaries are streamlines, so that along them

$$\psi = \text{constant} \quad \text{or} \quad \partial\varphi/\partial n = 0 \tag{2.1}$$

2. Boundaries of a Water Reservoir

Along boundaries of a water reservoir the pressure is determined by the hydrostatic law, e.g., along the segment $(-\infty, 0)$ of the x axis (Fig. 1):

$$p = p_0 + g\rho(H_1 - y) = p_0 + g\rho H_1 \tag{2.2}$$

By virtue of equation (1.4) along such a border

$$\varphi = -\kappa\left(\frac{p_0}{\rho g} + H_1\right) = \text{constant} \tag{2.3}$$

Thus, we have here a mixed problem of potential theory (a special case of Hilbert's problem): to find a harmonic function φ for a polygonal region, for which on some segments of the border region $\varphi = \text{constant}$, and on others $\partial\varphi/\partial n = 0$.

If the regions z and f are conformally mapped on the half plane of the auxiliary complex variable ζ, $f(\zeta)$ and $z(\zeta)$ can be found with the aid of the Christoffel-Schwarz formula.

If all the angles of the regions z and f are right angles or multiples of a right angle, elliptic or hyperelliptic integrals are obtained. Pavlovsky reviews in detail the results of the theory of elliptic functions which he needs for the solution of particular cases. His book achieved fame; a great many of his devices have been applied to the solution of numerous flow problems in hydrotechnical installations.

The soil water frequently moves in an anisotropic stratified soil, in which there exist two principal directions (parallel to the axes of filtration, x_1, y_1) along which the filtration coefficients are constants κ_1 and κ_2. Then the projections u, v of the filtration velocity upon the axis x_1, y_1 are described in the equations

$$u = \kappa_1 \frac{\partial\varphi}{\partial x_1}, \qquad v = \kappa_2 \frac{\partial\varphi}{\partial y_1}, \qquad \varphi = -\left(\frac{p}{\rho g} + x_1 \sin\alpha + y_1 \cos\alpha\right)$$

α being the angle of the x_1 axis with the horizontal axis. Both the stream function ψ, determined by the equations

$$u = \sqrt{\kappa_1\kappa_2}\,\frac{\partial\psi}{\partial y_1}, \qquad v = -\sqrt{\kappa_1\kappa_2}\,\frac{\partial\psi}{\partial x_1}$$

and the velocity potential satisfy the equation

$$\kappa_1 \frac{\partial^2\psi}{\partial x_1^{\,2}} + \kappa_2 \frac{\partial^2\psi}{\partial y_1^{\,2}} = 0$$

The change of variables $x_1 = \sqrt{\kappa_1}\,x$, $y_1 = \sqrt{\kappa_2}\,y$ leads to Laplace's equations for φ and ψ.

In the coordinate system x_1, y_1 we have a certain fictitious motion in a region obtained from the region x, y by an affine transformation [see Polubarinova-Kochina (131), Kozlov (70), Aravin (3)].

The following problem of the oblique cutoff is particularly interesting not only in the study of the flow around a vertical cutoff in an anisotropic

soil, but also in the problems of filtration around structures [Verigin (186)].

3. Problem of the Flow around an Oblique Cutoff (185)

The flow potential φ is determined within a constant term, which we can choose so that along AB in Fig. 2 $\varphi = -\frac{1}{2}\kappa H$; then along DE of Fig. 2 $\varphi = \frac{1}{2}\kappa H$, H being the effective head, $H = H_1 - H_2$.

We get from the Christoffel-Schwartz formula

$$z = A\int_0^\zeta \frac{(\zeta - a)d\zeta}{(\zeta+1)^{1-\alpha}(\zeta-1)^\alpha}, \qquad f = B\int_0^\zeta \frac{d\zeta}{\sqrt{1-\zeta^2}} = \frac{\kappa H}{\pi}\arc\sin\zeta \tag{2.4}$$

Since the boundaries of the region $ABCDE$ are straight lines passing through a common point, the integral for z must have a finite representa-

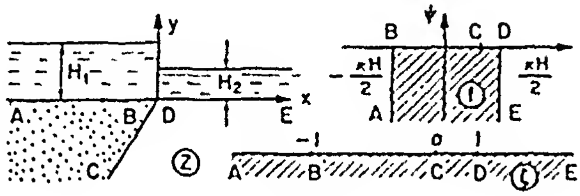


FIG. 2

tion. It is simpler to calculate it by using an appropriate substitution. For example, let

$$z = C(\zeta+1)^\beta(\zeta-1)^\gamma \tag{2.5}$$

and determine β and γ from the conditions:

$$\begin{array}{ll} \arg z = 0 & \text{along } DE \\ \arg z = -\pi + \pi\alpha & \text{along } DCB \\ \arg z = -\pi & \text{along } AB \end{array}$$

Thus it follows that C is real; since for $-1 < \zeta < 1$ we have

$$z = C(\zeta+1)^\beta(1-\zeta)^\gamma e^{-\pi i\gamma}$$

hence

$$\gamma = 1 - \alpha$$

since we have

$$z = C(-\zeta-1)^\beta(1-\zeta)^\gamma e^{-\pi i(\beta+\gamma)} \quad \text{for } -\infty < \zeta < -1,$$

we get $\beta + \gamma = 1$, or $\beta = \alpha$.

Consequently,

$$z = C(\zeta+1)^\alpha(\zeta-1)^{1-\alpha} \tag{2.6}$$

For $\zeta = a$, at the end of the cutoff, the velocity must become infinite, i.e., $dz/d\zeta = 0$. This results in the equation $a = 2\alpha - 1$. The condition that $z_0 = l_e^{-\pi i(1-\alpha)}$, where l is the length of the cutoff, determines C. Finally,

$$z = \frac{l(\zeta + 1)^{\alpha}(\zeta - 1)^{1-\alpha}}{2\alpha^{\alpha}(1 - \alpha)^{1-\alpha}}, \quad \zeta = \sin \frac{\pi f}{\kappa H} \tag{2.7}$$

[It is possible to determine the value of the first integral in the equation (2.4) in the form (2.5), to differentiate it, and then comparing it with (2.4), to determine β, γ, and α.]

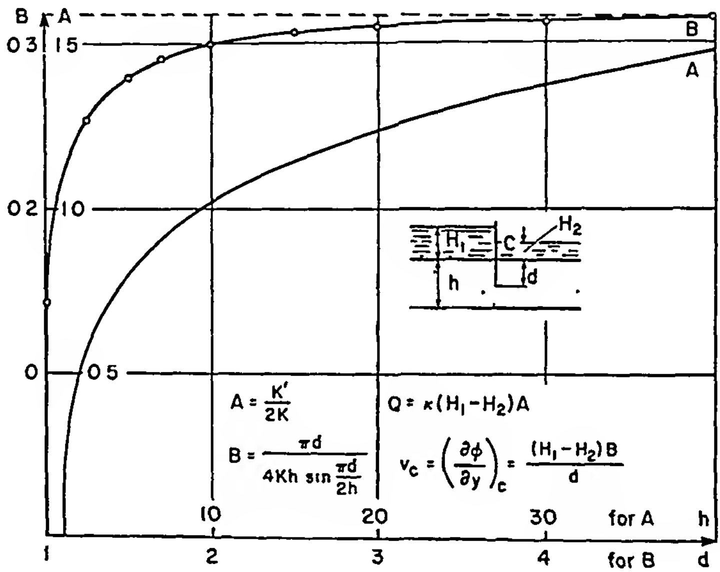


FIG. 3

For the equation of a vertical cutoff, at $\alpha = \frac{1}{2}$, we get

$$z = -h \cos \frac{\pi i}{\kappa H} \qquad \text{or} \qquad x = l \sin \frac{\pi \varphi}{\kappa H} \sinh \frac{\pi \psi}{\kappa H},$$

$$y = -l \cos \frac{\pi \varphi}{\kappa H} \cosh \frac{\pi \psi}{\kappa H}$$

The streamlines ψ = constant are ellipses, and the lines φ = constant are hyperbolas.

Without reproducing the formulas for a cutoff in a permeable layer of finite depth, we include a graph (Fig. 3) showing the relationship between discharge and the rate of discharge, as well as the ratio of the depth of the permeable layer to the length of the cutoff.

4. Filtration under the Apron

If the depth of the permeable base is h (Fig. 4), then the conformal mapping of the strip of the plane z and of the rectangle of the plane f

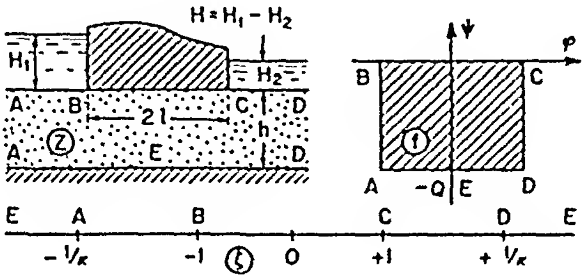


FIG. 4

upon the lower half-plane ζ gives us ($2l$ is the width of the apron)

$$f = \frac{\kappa H}{2K}\int_0^\zeta \frac{d\zeta}{\sqrt{(1-\zeta^2)(1-k^2\zeta^2)}} = \frac{\kappa H}{2K}F(k,\zeta)$$

$$z = \frac{h}{\pi}\log\frac{1-k\zeta}{1+k\zeta} = \frac{h}{\pi}\log\frac{1-k\,(sn)\,u}{1+k\,(sn)\,u} \tag{2.8}$$

$$u = \frac{2Kf}{\kappa H}, \qquad k = \tanh\frac{\pi l}{2h}$$

$$K = \int_0^{\frac{\pi}{2}} \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}}$$

The total discharge in the flow around the apron is

$$Q = \frac{\kappa H}{2}\frac{K'}{K} \quad (K' = K(k'), \quad K' = \sqrt{1-k^2})$$

Values of $Q' = Q/\kappa H$ as a function of l/h

l/h	0.4	1	1.122	2	3	4	5	6
Q'	0.811	0.533	0.500	0.300	0.258	0.204	0.170	0.146
l/h	7	7.5	8	10	12	14	15	16
Q'	0.126	0.119	0.113	0.0919	0.0776	0.0672	0.0630	0.0592
l/h	20	25	30	40	50			
Q'	0.0479	0.0386	0.0354	0.0245	0.0197			

If $h \to \infty$, $Q \to \infty$, and $\kappa \to 0$, one can take the region of motion in the plane z, instead of the lower half-plane ζ, and map upon it the half-strip of the plane f.

We get then

$$f = \frac{\kappa H}{2K}\,\text{arc sin}\,\frac{z}{l} \quad \text{or } z = l \sin\frac{2Kf}{\kappa H} \tag{2.9}$$

There the flow lines are ellipses, and the lines φ = constant are confocal hyperbolas.

Meleshchenko (83) considered the apron at an oblique impermeable lower boundary; Bazanov, the flow around a similar broken one (filtration under jumps); Girinsky (40) solved the simplest problems for the case of a constant head on the soil bottom.

For an apron with cutoffs, the solution becomes more complicated, because the presence of each cutoff increases the number of angles of the flow region by three, and, consequently, introduces three parameters, i.e., the complex numbers correspond to the new apexes in the ζ – plane. The calculation of these parameters is very difficult. Sometimes a series of calculations is made, assuming values of the parameters and determining from them the lengths of the segments bounding the flow region. Kozlov (68,69,71,72) considered systems with many cutoffs. More complicated systems have been studied by Segal (151,153). He applied the following approximate method of conformal mapping. First he omitted from the flow region one or several segments and mapped the remaining simpler polygon onto an auxiliary semi-plane. Thereby the omitted segments were mapped into certain curvilinear contours. If those contours were nearby rectilinear, they were replaced by rectilinear segments, and a further mapping was made onto the final semiplane.

Pavlovsky (115) proposed an approximation method of fragments consisting of the following. At small depths of the permeable layer the flow region is divided into sections by straight lines which are continuations of the cutoffs, and these verticals are taken to be lines of constant potential (see end of section X). In a paper by Zamarin (189), and in his paper written jointly with Shipenko (190), there are given nomograms for such calculations. Other approximate methods, especially one based on the application of finite differences, are presented in Meleshchenko's paper (85); the calculation methods for an infinitely deep layer are discussed by Kozlov (73) and Meleshchenko.

We note that in the foreign literature there exists no such systematic discussion of solutions for filtration problems under hydrotechnic structures. Particular problems have been solved by single authors. Rossbach (150) investigated systems with many cutoffs using the following approximate method: he repeatedly applied the Koebe transformation to an intermediate region representing the map of the given region without the cutoff upon a half-plane. The Koebe transformation, as a procedure of an approximate conformal mapping of a region upon a circle, gives a very slow convergent process.

Pavlovsky's paper attracted the attention of N. E. Zhukovsky, who, in the nineteen twenties, returned to the filtration theory. A paper

published by him in 1923 is partly devoted to the solution of the same problems as that of Pavlovsky but by a different method, namely that of generating and directing nets, which he developed earlier in connection with the theory of jets. The essence of his method consists in the construction of functions from their singularities, the geometric illustration playing a secondary role.

III. Motions with a Free Surface—Zhukovsky's Function

The second part of Zhukovsky's paper (195) is devoted to the motion of soil water with formation of a free surface (depression or desiccation surface). Since the pressure on this surface must be constant and equal to the atmospheric pressure, on the free surface, according to equation (1.4), there must be satisfied the condition

$$\varphi + \kappa y = \text{constant} \tag{3.1}$$

Furthermore, the line of the free surface (a plane problem is considered here) is a streamline, and hence along it

$$\psi = \text{constant} \tag{3.2}$$

Zhukovsky introduced the function

$$\theta = \theta_1 + i\theta_2 = f - i\kappa z = \varphi + \kappa y + i(\psi - \kappa x) \tag{3.3}$$

The conditions (3.1) and (3.2) can also be written in the form

$$\operatorname{Im} f = \text{constant}, \qquad \operatorname{Re} \theta = \text{constant}$$

If the impermeable walls are vertical segments and the boundaries of the water store horizontal ones, then along them we have respectively: $\operatorname{Im} \theta = \text{constant}$, $\operatorname{Im} f = \text{constant}$, and $\operatorname{Re} \theta = \text{constant}$, $\operatorname{Re} f = \text{constant}$. This permits us to solve a series of filtration problems with free surface by the foregoing methods.

1. *Zhukovsky's Problem—the Flow around Cutoffs (Fig. 5)*

Let d be the length of the cutoff BC and d_1 the length of the segment BD. In the plane f we get an angle, in the plane θ a half-strip which are mapped onto the lower semi-plane ζ, respectively, by means of the relations

$$f = \frac{\kappa(d_1 + H)}{\sqrt{2}} \sqrt{\zeta + 1} - \kappa H, \qquad \theta = -\frac{\kappa H}{\pi} \operatorname{arc\,cos} \zeta$$

Hence we get the relations between f, θ, and z:

$$f = -\kappa\left[H + (d_1 + H)\cos\frac{\pi\theta}{2\kappa H}\right]$$

$$z = -\frac{if}{\kappa} - \frac{2iH}{\pi}\arc\cos\frac{f + \kappa H}{\kappa(d_1 + H)}$$
$$= i\left[\frac{\theta}{\kappa} + H - (d_1 + H)\cos\frac{\pi\theta}{2\kappa H}\right] \quad (3.4)$$

At the end-point C of the cutoff the velocity must be infinite, which permits us to find f_c (or $\zeta_c = a$), and consequently, the length of the cutoff, as

$$d = -H + \sqrt{(d_1 + H)^2 - \frac{4H^2}{\pi^2}} + \frac{2H}{\pi}\arc\sin\frac{2H}{\pi(d_1 + H)} \quad (3.5)$$

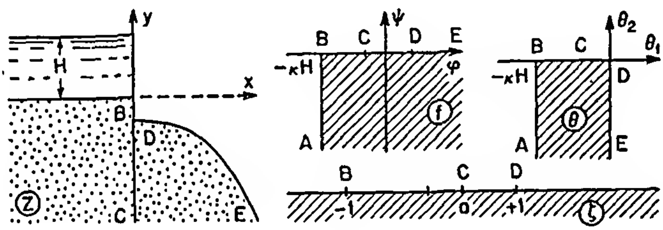


FIG. 5

Along the free surface, there obtains θ_1 = constant = 0, and we get the equation of the free surface,

$$y = H - (d_1 + H)ch\frac{\pi x}{2H} \quad (3.6)$$

Vedernikov (172), who treated this problem, taking into account also the capillarity, gives the relation between d/H and d_1/H by means of the following tabulation:

$\frac{d}{H} =$	0	0.074	0.213	0.64	1.10	2.066
$\frac{d_1}{H} =$	−0.363	−0.2	0	0.5	1	2
$\frac{h}{H} = \frac{d - d_1}{H} =$	0.363	0.274	0.213	0.14	0.10	0.066

The negative values of d_1/H indicate that the water at the point D rises higher than at the point B.

Zhukovsky, having solved the problem by the method of generating and directing nets, made an error, which has been noted by many authors, in the interpretation of the solution. He assumed that the points

C and D coincide and that the velocity at the end of the cutoff is equal to zero.

The second problem of Zhukovsky, concerning the inflow to a drain gap, results from the previous one by a rotation of the z-plane through 90° and an interchange of φ and ψ. (We shall return to this problem in section V.)

Nelson-Skornyakov (99) generalized the Zhukovsky problem on the flow around a cutoff by placing in the lower part of the flow a vertical segment which extends downwards into infinity. Then there exist among the streamlines parabolic-shaped curves extending downward toward infinity, each of which can be taken as an impermeable boundary. An analogous method, the artificial introduction of a semi-infinite cutoff, is applied by the author to another model (see section IX).

Aravin (4) solved the problem of the inflow of water from infinity, with formation of free surface, to a reservoir enclosed by two cutoffs of equal length. The flow net is shown for a special case (width of channel 30 m, depth of cutoffs 4.34 m, interval of seepage 3 m).

IV. Motion with a Free Surface. Velocity Hodograph

Differentiating the equation $\varphi + \kappa y =$ constant with respect to the arc s of the free surface, and multiplying each term by $\partial\varphi/\partial s$, we get

$$\left(\frac{\partial \varphi}{\partial s}\right)^2 + \kappa \frac{\partial \varphi}{\partial s}\frac{dy}{ds} = 0 \quad \text{or} \quad v_x^2 + v_y^2 + \kappa v_y = 0 \tag{4.1}$$

In the plane v_x, v_y this is the equation of a circle. Along the rigid walls the velocity vector is parallel to these walls, and, consequently, the tip of this vector will describe in the plane v_x, v_y a straight line which passes through the origin and is parallel to the wall. Along a rectilinear water boundary the velocity vector is perpendicular to the boundary, therefore the velocity hodograph is a straight line through the origin and perpendicular to the boundary of the water system.

If the hodograph consists only of straight lines and a circumference all of which pass through the origin of the coordinates, the transformation by inverse radii with the center at the origin changes the contour into a polygon bounded by straight lines, and consequently, it is possible again to apply the Christoffel-Schwarz formula. The first to apply this method was Vedernikov (165). He investigated the filtration of water from channels at triangular and trapozoidal cross sections, and also a series of other flows with a free surface (see section VI). Also to Vedernikov goes the credit for numerous investigations of the influence of the capillary rise upon the filtration of liquids. The influence of capillarity is taken

into account by assuming that the pressure on the free surface is equal to the atmospheric pressure changed in accordance with the capillary rise h_k [see for example Zhukovsky (197)]

$$\varphi + \kappa y = -\kappa\left(\frac{p_0}{\rho g} - h_k\right)$$

or, at $p_0 = 0$

$$\varphi + \kappa y = \kappa h_k \tag{4.2}$$

In the case of anisotropic soil, we get at the free surface the equation

$$\varphi + \sqrt{\kappa_1}\, x \sin \alpha + \sqrt{\kappa_2}\, y \cos \alpha = \text{constant}$$

in the v_x, v_y-plane, to the free surface there corresponds the circle

$$v_x^2 + v_y^2 + \sqrt{\kappa_1} \sin \alpha v_x + \sqrt{\kappa_2} \cos \alpha v_y = 0.$$

For a case deviating from the Darcy law, Khristianovich (65) gets equations which differ from those of a circle

$$\phi(v) + \sin \theta = 0 \quad \text{or} \quad v\phi(v) + v_y = 0$$

As a first example of application of the velocity hodograph we quote Risenkampf's (146) solution of a problem treated earlier by Vedernikov (165).

1. Filtration from a Channel of a Trapezoidal Cross Section (Fig. 6)

If we put

$$u = \frac{dz}{df} = \frac{1}{v_x - iv_y}$$

the triangle in the u-plane represents the inversion of the region of the $v_x + iv_y$-plane with respect to a circle with radius l having its center at the origin of coordinates. We have

$$du = -\frac{i}{\kappa}\frac{A\,d\zeta}{\zeta^{1/2+\alpha}(\zeta - 1)^{1-\alpha}} \left(A = \frac{\Gamma\frac{1}{2}(+\alpha)}{\sqrt{\pi}\,\Gamma(\alpha)}\right) \tag{4.3}$$

The mapping of the half-strip f onto the semi-plane ζ yields

$$\zeta = \frac{\sin^2 w}{\sin^2 \theta}\left(w = \frac{\pi f}{2Qi}, \quad \zeta = \frac{\pi Q'}{2Q}, \quad \zeta_D = d = \frac{1}{\sin^2 \theta}\right) \tag{4.4}$$

where Q is the full discharge through the section DCB, and Q' is the discharge through both slopes BC and $B'C'$.

Substituting (4.4) into (4.3), putting $t = e^{iw}$ and developing the right-hand member of the equation for du into a series of powers of t, we get

$$du = \frac{4A}{\kappa} \sin \theta(1 + a_1 t^2 + a_2 t^4 + \cdots)dt \tag{4.5}$$

where

$$\begin{aligned} a_1 &= 1 + 2\cos 2\theta + 2(2 - \cos 2\theta)\alpha = 3 - 4(1 - \alpha)\sin^2 \theta \\ a_2 &= a_1^2 + 2a_1 \cos 2\theta - 3 - 2\cos 2\theta + 4(1 - \cos 2\theta)\alpha \end{aligned} \tag{4.6}$$

. . . .

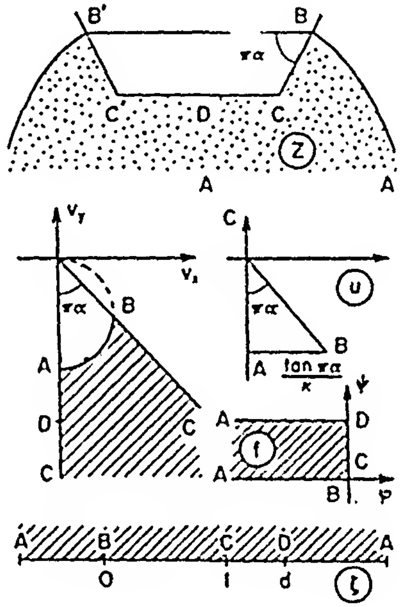


Fig. 6

Integrating the equation (4.5) with respect to t, and reintroducing the variable w, we get after a second integration

$$z = -\frac{if}{\kappa} + \frac{8Q}{\kappa\pi\sqrt{\pi}} \frac{\Gamma(\alpha + \frac{1}{2})}{\Gamma(\alpha)} \sin \theta \left(e^{iw} + \frac{a_1}{3^2} e^{3iw} + \frac{a_2}{5^2} e^{5iw} + \cdots \right)$$

The condition $z_B - z_D = \frac{1}{2}B + iH$ leads to:

$$\begin{aligned} B &= \frac{Q}{\kappa}\left[1 - \frac{8A}{\pi} \sin \theta \left(1 + \frac{a_1}{3^2} + \frac{a_2}{5^2} + \cdots \right)\right] \\ H &= \frac{Q}{\kappa} \frac{4A}{\pi} \sin \theta \left(1 - \frac{a_1}{3^2} + \frac{a_2}{5^2} - \cdots \right) \end{aligned} \tag{4.7}$$

Hence

$$Q = \kappa\mu(B + 2H) \quad \left(\mu = \left\{1 - \frac{16A}{\pi} \sin \theta \left[\frac{a_1}{3^2} + \frac{a_3}{7^2} + \cdots \right]\right\}^{-1} \right) \tag{4.8}$$

Risenkampf (146) gives the approximate, but sufficiently accurate formulas:

$$\sin\theta = \frac{1}{C(1 + B/2H)}, \quad C = \frac{8\Gamma(\frac{1}{2} + \alpha)}{\pi\sqrt{\pi}\,\Gamma(\alpha)} \left(\theta = \frac{\pi Q'}{2Q}\right) \tag{4.9}$$

$$\frac{1}{\mu} = 1 - \frac{2}{3(1 + B/2H)} + \frac{8(1 - \alpha)}{9C^2(1 + B/2H)^3}.$$

A table of values of C in relation to the values of the angle $\beta = \pi\alpha$ (in degrees) follows.

$\beta = \pi\alpha =$	0°	3.6	7.2	10.8	14.4	18.0	21.6
$C =$	0	0.050	0.097	0.141	0.184	0.225	0.264
$\beta = \pi\alpha =$	25.2	28.8	32.4	36.0	39.6	43.2	46.8
$C =$	0.302	0.338	0.373	0.406	0.439	0.470	0.501
$\beta = \pi\alpha =$	50.4						
$C =$	0.530						

Vedernikov (172) carried out calculations for trapezoidal, rectangular beds; in particular, the velocity distribution along the perimeter is calculated for several examples.

We note that in the problem considered the velocity at infinity is equal to the filtration coefficient κ. Vedernikov (165) and Risenkampf (147) considered also examples of filtration from channels (or water reservoirs) with the velocity at infinity equal to zero. As Risenkampf points out, flow lines with points of inflection are obtained.

As a second example, we present Falkovich's solution of a problem, studied by many authors: Nelson-Skornyakov (93), Risenkampf (145), Numerov (109), and others.

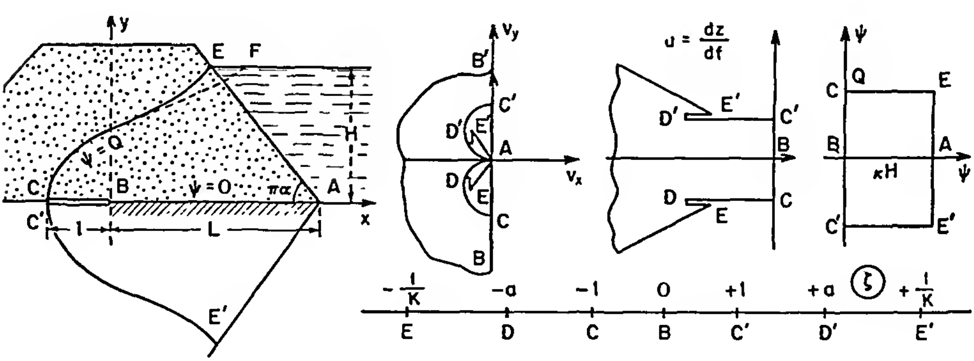


FIG. 7

2. Motion of Soil Water through a Dam Drained at the Base

Using the analytic continuation of the flow across the base line AB, the author obtains the solution in a symmetric form and expresses it by means of elliptic functions.

We state the final results. The relation between z and $w = if/\kappa$ is:

$$
\begin{aligned}
z - l &= A\left[w\int_0^w \frac{sn^2w_1 - a^2}{dn^{2\alpha}w_1}\,dw_1 - \int_0^w w_1\,\frac{sn^2w_1 - a^2}{dn^{2\alpha}w_1}\,dw_1\right] \\
w &= \frac{if}{\kappa} = \frac{i(\varphi + i\psi)}{\kappa}, \qquad A = \frac{J_4 - k'^{2\alpha}J_2 \tan \pi\alpha}{J_2J_3 - J_1J_4} \\
a^2 &= \frac{J_3 - k'^{2\alpha}J_1 \tan \pi\alpha}{J_4 - k'^{2\alpha}J_2 \tan \pi\alpha}, \qquad J_1 = \int_0^{\frac{1}{2}\pi} \frac{\sin^2\varphi\,d\varphi}{(1 - k^2\sin^2\varphi)^{\frac{1}{2}+\alpha}} \qquad (4.10) \\
J_3 &= \int_0^{\frac{1}{2}\pi} \frac{d\varphi}{\cos^{2\alpha}\varphi(1 - k'^2\sin^2\varphi)^{\frac{1}{2}-\alpha}}, \qquad J_2 = \int_0^{\frac{1}{2}\pi} \frac{d\varphi}{(1 - k^2\sin^2\varphi)^{\frac{1}{2}+\alpha}} \\
J_4 &= \int_0^{\frac{1}{2}\pi} \frac{d\varphi}{\cos^{2\alpha}\varphi(1 - k'^2\sin^2\varphi)^{\frac{1}{2}-\alpha}}
\end{aligned}
$$

The discharge Q and the lengths of the segments l and L are calculated by means of the formulas

$$
\begin{aligned}
Q &= \frac{\kappa HK}{K'}, \qquad l = \frac{HK}{K'} + \frac{A}{K}\int_0^{\frac{1}{2}\pi} \frac{F(\varphi,k)(\sin^2\varphi - a^2)}{(1 - k^2\sin^2\varphi)^{\alpha+\frac{1}{2}}}\,d\varphi \\
L &= A\left[H\int_0^{\frac{1}{2}\pi} \frac{(\sin^2\varphi + a^2\cos^2\varphi)d\varphi}{\cos^{2-2\alpha}\varphi(1 - k'^2\sin^2\varphi)^{\alpha+\frac{1}{2}}}\right. \\
&\qquad \left. - \int_0^{\frac{1}{2}\pi} \frac{F(k',\varphi)(\sin^2\varphi + a^2\cos^2\varphi)d\varphi}{\cos^{2-2\alpha}\varphi(1 - k'^2\sin^2\varphi)^{\alpha+\frac{1}{2}}}\right] \\
K &= \int_0^{\frac{1}{2}\pi} \frac{d\varphi}{\sqrt{1 - k^2\sin^2\varphi}}, \qquad K' = K(k'), \qquad k' = \sqrt{1 - k^2} \qquad (4.11)
\end{aligned}
$$

In the limiting case $\alpha = 0$, i.e., when the slope of the dam is nearly horizontal, the formulas are simplified as follows:

$$
\begin{aligned}
z - l &= -\frac{\kappa w^2}{2Q} - \frac{2H}{\pi}\log\frac{\theta(w)}{\theta(0)} \\
l &= \frac{HK}{K'} + \frac{H}{\pi}\log\frac{1}{k'} \\
\theta(w) &= 1 - 2q\cos\frac{\pi w}{K} + 2q^4\cos\frac{2\pi w}{K} - 2q^9\cos\frac{3\pi w}{K} + \cdots \qquad (4.12) \\
L &= l + \frac{H}{\pi}\log\frac{1}{k'}, \quad q = \exp\frac{-\pi k'}{k} \\
Q &= \frac{\kappa HK}{K'}, \quad k = \exp\frac{-\pi(L + l)}{H}
\end{aligned}
$$

For another limiting case $\alpha = \frac{1}{2}$, i.e., for a dam with a vertical slope, Falkovich (31) obtains

$$
\begin{aligned}
z - l &= \frac{2Hi}{\pi} \int_0^w \log \frac{cnw + ik'snw}{dnw}\, dw \\
l &= \frac{2H}{\pi} \int_0^k \arctan\left(k' \frac{snw}{cnw}\right) dw = \frac{2HK}{\pi} \int_0^{\frac{1}{2}\pi} \frac{F(\varphi,k)d\varphi}{1 - k'^2 sn^2\varphi} \\
L &= \frac{H}{\pi}\left[\log\frac{1+k'}{1-k'} - 2k' \int_0^{\frac{1}{2}\pi} \frac{F(\varphi,k')}{K'} \frac{\cos\varphi d\varphi}{1 - k'^2\sin^2\varphi}\right]
\end{aligned}
\tag{4.13}
$$

We give Risenkampf's (147) results concerning this problem. When the upstream slope EA moves to infinity, the free surface becomes a parabola (the filtration coefficient is taken equal to unity),

$$
\frac{y^2}{2Q} = x + \frac{Q}{2} \tag{4.14}
$$

The relation between z and f is

$$
z = -\frac{w^2}{2Q} \tag{4.15}
$$

Assuming $z = -w^2/2Q + \phi(w)$, Risenkampf gets for z and for the free surface the expressions

$$
\begin{gathered}
z = -\frac{w^2}{2Q} + 2a_1\left(\cos\frac{\pi w}{Q} - 1\right) + \cdots \\
2a_1 \approx \frac{H}{\pi} \frac{1 - \frac{Q}{H}\tan\pi\alpha}{\sinh\frac{\pi H}{Q}} \\
x + \frac{Q}{2} = \frac{y^2}{2Q} - 2\alpha_1\left(\cosh\frac{\pi y}{Q} + 1\right) + \cdots
\end{gathered}
\tag{4.16}
$$

The author gives the following approximate formulas.

$$
\begin{aligned}
l &\approx \frac{Q}{2} \approx \frac{H^2}{4(L+\lambda)} \\
\lambda &= \frac{H}{\pi}\left[1 - \frac{\tan\pi\alpha}{2(L/H + 1/\pi)}\right]
\end{aligned}
\tag{4.17}
$$

The angle $\pi\alpha$ is assumed sufficiently small here, namely, $\tan \pi\alpha < \kappa H/Q$. Risenkampf gives the following approximate method for the construction of the depression line (Fig. 7). From the point E of the head water, along the water level, lay off the segment $EF = \lambda$, and construct the parabola (4.14) of focus B passing through the point F; then, through the point E and perpendicular to the slope draw a smooth curve continuing the parabola.

Numerov (104) solved the same problem by another method (see section X). For a dam with a vertical slope the author gets the approximate formula

$$Q \approx \frac{\kappa H^2}{L + \sqrt{L^2 + H^2/3}} \tag{4.18}$$

Nelson-Skornyakov (95) investigated special cases of this problem for $\alpha = \frac{1}{2}$ and $\alpha = 0$.

Problems with drain pipes have been investigated by many authors. Grib (48) investigated the flow in the case where the free surface is in a region bounded by a waterproof wedge-shaped layer, and with a drain pipe present. Vedernikov (171) investigated the case of an infinite series of drains under the free surface, taking into account the capillarity of the soil; Khomovskaya (63) and Vedernikov (172), using a different method, investigated the problem of one drain in an infinite region (with a free surface). Nelson-Skornyakov (103) investigated the generalization of Slichter's problem of the inflow of soil waters to drain channels at the boundary of the body of permeable soil. Falkovich (31) investigated this problem by a simpler method.

We note that in all applications, the construction of the hydrodynamic flow net is important. Girinsky (40–42) and Zamarin (189) [see also V. Fandeev (32)] worked out practical devices for this purpose.

V. The Seepage Line

Observations and experiments have established that when soil water flows into a well, the level of the free surface is higher than the water level in the well, so that part of the water leaks directly into the air. The interface between the air and the soil water is called the "seepage segment." Along it the pressure is constant (equal to atmospheric). According to (1.4),

$$-\frac{\kappa p}{\rho g} = \varphi + \kappa y \tag{5.1}$$

Therefore, on the seepage segment

$$\varphi + \kappa y = \text{constant} \tag{5.2}$$

or, using the Zhukovsky function $\theta = f - i\kappa z$, we get

$$\operatorname{Re} \theta = \text{constant} \tag{5.3}$$

Now differentiate (5.2) along the seepage segment. Denoting by α its angle with the axis of abscissas, we get

$$\frac{\partial \varphi}{\partial s} = v_x \cos \alpha + v_y \sin \alpha, \quad \frac{dy}{ds} = \sin \alpha$$

and, therefore, along the line, we have

$$v_x \cos \alpha + v_y \sin \alpha + \kappa \sin \alpha = 0 \tag{5.4}$$

In the plane v_x, v_y the equation (5.4) determines a straight line through the point $(0, -\kappa)$ and perpendicular to the seepage line. Along the seepage segment, for an anisotropic soil the equation of the straight line is

$$\frac{v_x}{\sqrt{\kappa_1}} \cos \alpha + \frac{v_y}{\sqrt{\kappa_2}} \sin \alpha + \sin \alpha = 0$$

For a flow not following Darcy's law, Khristianovich (65) obtains the equation

$$\phi(v) \cos (\theta - \alpha) + \sin \alpha = 0$$

where α is the angle of the rectilinear seepage segment with the x-axis. For a first example of a seepage segment Davison (29) examined the following problem.

1. *A Triangular Earth Dam Filled with Water*

A triangular earth dam filled with water is situated on a horizontal impermeable base. The flow domain is a triangle, the region in the plane

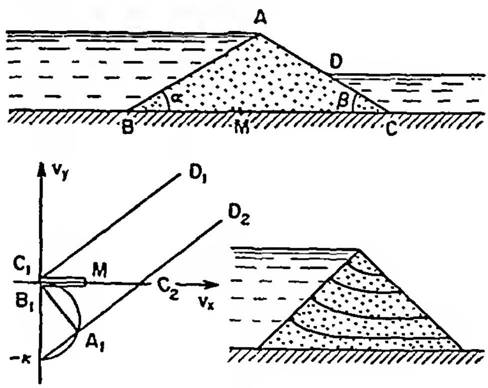


FIG. 8

$v_x v_y$ a polygon so that both domains are mapped on the half-plane by means of the Christoffel-Schwartz formula.

If there is no tail water, the line C_1D_1 vanishes from the velocity hodograph, and we get in the v_x, v_y-plane the triangle $A_1B_1C_2$. If the slopes make 45° angles with the horizontal, the triangle $A_1B_1C_2$ is similar to triangle ABC, and we get

$$w = v_x - iv_y = az + b$$

By determining a and b_1, we get

$$w = \frac{\kappa}{2}\left(\frac{z}{H_1} + 1 + i\right)$$

Hence integrating, we obtain

$$f = \frac{\kappa}{2}\left[\frac{z^2}{2H_1} + z(1 + i)\right] + \text{constant}$$

The streamlines are hyperbolas $(x + H_1)(y + H_1) = \text{constant}$, orthogonal to AB and asymptotic to BC. The discharge is $3\kappa H_1/2$. Nelson-Skornyakov investigated this case.

As a second example let us consider a problem of Vedernikov (172). The solution given below belongs to Falkovich.

2. Drain Channel of Trapezoidal Cross Section

Here all the boundaries of the velocity hodograph intersect in the point $M(0, -\alpha)$. Performing an inversion with respect to a circle centered

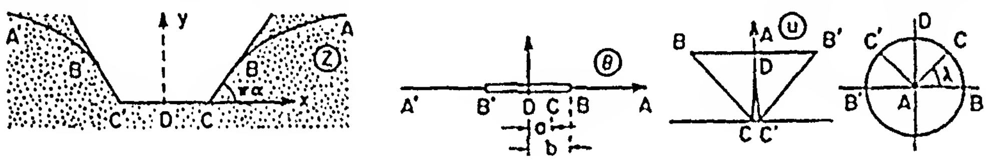


FIG. 9

at M, we obtain in the plane $u = [v_x - i(v_y + \kappa)]^{-1}$ a rectilinear triangle with a slit (Fig. 9). Because the condition (5.3) is satisfied along the entire flow contour, the region θ is the lower half-plane. Using the Christoffel-Schwartz formula, we get

$$du = \frac{A\theta d\theta}{(\theta^2 - a^2)^{1-\alpha}(\theta^2 - b^2)^{\alpha+\frac{1}{2}}}, \quad \left(A = \frac{2\sqrt{\pi}\sqrt{b^2 - a^2}}{\cos \pi\alpha\Gamma(\alpha)\Gamma(\frac{1}{2} - \alpha)}\right)$$

Continuing the flow analytically across the free surface, we obtain in the θ-plane a region bounded by a slit along the real axis; mapping the outside of the slit on the inside of the unit circle in the t-plane (Fig. 9), we get $\theta = \frac{1}{2}b(t + t^{-1})$, and hence

$$du = \frac{2A}{b}\frac{(1 + t^2)dt}{(1 - t^2)^{2\alpha}(t^4 - 2t^2 \cos 2\lambda + 1)^{1-\alpha}}, \quad a = b\cos\lambda \tag{5.5}$$

Developing the right-hand term in a series of powers of t, we get

$$du = \frac{2A}{b}(1 + a_1t^2 + a_2t^4 + \cdots)dt \tag{5.6}$$

where the coefficients $a_1, a_2, \cdots$ are determined by equation (4.6).

Integrating (5.6) with respect to t, we get

$$u - \frac{i}{\kappa} = \frac{2A}{b}\left(t + \frac{a_1}{3}t^3 + \frac{a_2}{5}t^5 + \cdots\right) \tag{5.7}$$

By applying this equality to the point B, we obtain the equation

$$-\sin \alpha\pi = \frac{4\sqrt{\pi}\sin\lambda}{\Gamma(\alpha)\Gamma(\frac{1}{2}-\alpha)}\left(1 + \frac{a_1}{3} + \frac{a_2}{5} + \cdots\right) \tag{5.8}$$

which determines λ.

The equation (5.7) can be given the form

$$-\frac{i}{\kappa}(d\varphi + id\varphi) = A\left(t + \frac{a_1}{3}t^3 + \frac{a_2}{5}t^5 + \cdots\right)\left(1 - \frac{1}{t^2}\right)dt$$

By integrating along BCD, the real part of this expression, we get

$$\begin{aligned}\psi = \kappa A\Bigg[\frac{1}{1.2}\cos 2v + \left(\frac{a_1}{3.4}\cos 4v + \frac{a_2}{5.6}\cos 6v + \cdots\right) \\ - \left(\frac{a_1}{2.3}\cos 2v + \frac{a_2}{4.5}\cos 4v + \cdots\right) + \left(\frac{1}{2} - \frac{a_1}{3.4} + \frac{a_2}{5.6}\cdots\right) \\ - \left(\frac{a_1}{2.3} - \frac{a_2}{4.5} + \cdots\right)\Bigg]\end{aligned}$$

Hence, finally,

$$Q = \frac{4\kappa b\sqrt{\pi}\sin\lambda}{\cos\alpha\pi\Gamma(\alpha)\Gamma(\frac{1}{2}-\alpha)}\left(1 - \frac{a_1}{1.3} + \frac{a_2}{3.5} - \cdots\right) \tag{5.9}$$

By substituting in (5.9) the value

$$b = \tfrac{1}{2}[Q - (B + 2H\cot\pi\alpha)]$$

we get the formula for the quantity of liquid entering the channel:

$$Q = \frac{B + 2H\cot\pi\alpha}{2 - \cos\pi\alpha\Gamma(\alpha)\Gamma(\frac{1}{2}-\alpha)/\sqrt{\pi}\sin\lambda M(\lambda)}$$

$$M(\lambda) = 1 - \frac{a_1}{1.3} + \frac{a_2}{3.5} - \cdots \tag{5.10}$$

The following special cases are obtained: (1) a channel of rectangular cross section; (2) the Zhukovsky drain slit (Fig. 10), for which the equation of the free surface is

$$x = \frac{1}{2}\left[\left(b + \frac{Q}{\kappa}\right)\cosh\frac{\kappa\pi y}{Q} - \frac{Q}{\kappa}\right] \tag{5.11}$$

and for which the width a of the seepage interval is connected with the width b of the channel and the discharge Q by the relation

$$a = -\frac{Q}{\kappa\pi} \operatorname{arc\,cos} \frac{2Q}{\pi(Q + \kappa b)} - \frac{b}{2} + \frac{1}{2}\left(\frac{Q}{\kappa} + b\right)\sqrt{1 - \frac{4Q^2}{\pi^2(\kappa b + Q)^2}} \qquad (5.12)$$

Some generalizations of the problem on the inflow to a drain slot are given by Nelson-Skornyakov (103). (3) A special case of a rectangular cross section is a vertical slot (with $b = 0$). Vedernikov constructs for it a system of isobars, flow lines, and equipotential lines. The height of the exit orifice is connected with the discharge by the formula

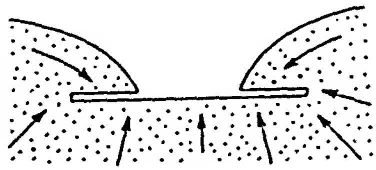


FIG. 10

$$h = \frac{Q}{\pi\kappa} 2 \log 2 = 0.441 \frac{Q}{\kappa} \qquad (5.13)$$

Assuming one of the equipotential lines as the contour, the author obtains the inflow to a drain filled with water.

VI. General Case of the Velocity Hodograph

In the general case, when the boundary of the flow domain contains both the free surface and the seepage line, the velocity hodograph consists of a circumference and of straight lines which have no point in common; the problem of conformal mapping of such a circular polygon cannot be reduced to the application of the Christoffel-Schwartz formula. The same is true of the case of evaporation from the free surface or of infiltration in the surface; it is then assumed that the discharge of the liquid across any part of the surface is proportional to the difference of the abscissas of the ends of this part, $\psi - \psi_0 = c(x_0 - x)$, where c is the quantity of water which evaporates per unit time and per unit length of the horizontal projection of the arc of the free surface. Thus, on the free surface,

$$\psi + cx = \text{constant} \qquad (6.1)$$

Differentiating this equation along the free surface, we get

$$v_x \sin\alpha - v_y \cos\alpha + c \cos\alpha = 0$$

where α is the angle between the tangent to the depression curve and the x-axis. Eliminating the variable parameter α from this equation and the equation (5.4) derived for the free surface, we get the equation of a circle

$$v_x^2 + v_y^2 + (\kappa - c)v_y = \kappa c \qquad (6.2)$$

the circle has a radius of $\frac{1}{2}(\kappa + c)$, its center is at the point $(0,\frac{1}{2}(c - \kappa))$, and it goes through the points $(0,c)$ and $(0,-\kappa)$.

Davison (19) showed that if the given boundaries are rectilinear, each of their segments satisfies two conditions of the type

$$ax + by + c\varphi + g\psi = h$$

with constant coefficients. In terms of the complex numbers k_i, l_i, m_i, n_i, these conditions can be written as follows:

$$\operatorname{Im}\,(k_iz + l_if) = p_i, \qquad \operatorname{Im}\,(m_iz + n_if) = q_i, \qquad \left(\begin{vmatrix} k_i l_i \\ m_i n_i \end{vmatrix}\right) \neq 0 \quad (6.3)$$

where p_i and q_i are real numbers, and i is the serial number of the contour section.

Let us assume that the flow domain is mapped on the upper semi-plane of the auxiliary complex variable ζ; to the boundaries of the flow domain there are corresponding segments of the real axis. Differentiating along this axis we get

$$\operatorname{Im}\,(k_iZ + l_iF) = 0, \qquad \operatorname{Im}\,(m_iZ + n_iF) = 0, \qquad (Z = dz/d\zeta,\ F = df/d\zeta) \quad (6.4)$$

Hence, by division

$$\operatorname{Im}\frac{k_i + l_iw}{m_i + n_iw} = 0, \quad \left(w = v_x - iv_y = \frac{df}{dz} = \frac{F}{Z}\right) \qquad (6.5)$$

This is the equation of a circle or a straight line in the plane of the complex velocity w.

Differentiating (6.5) Davison (19,20,21) obtains

$$\arg\frac{dz}{dw} + 3\arg\,(m_i + n_iw) - \arg\,(m_il_i - n_ik_i) = n\pi$$

where n is an integer. He used this relation in 1932 to solve the problem of filtration through an earth dam.

In 1934, Hamel (49) gave an analogous solution, using a mapping onto a circle instead of onto a semi-plane. By this method several examples have been calculated. Polubarinova-Kochina (123–130,136) gave in 1938 a simpler solution, using the method of section VII, and made then extensive calculations. Risenkampf (146) gave some supplements to this solution and investigated in detail some special cases of the problem.

VII. Application of the Analytic Theory of Differential Equations

The process of finding the complex velocity w as a function of ζ consists essentially of a conformal mapping of a circular polygon with n

apexes, onto a semi-plane. Let the angles at the apexes, which in the ζ-plane become points $a,b, \cdots c$ of the real axis, be respectively $\pi\alpha$, $\pi\beta, \cdots, \pi\gamma$; and let the angle at the apex, which becomes the infinitely remote point

$$(\zeta = \infty), \text{ be } \pi(\delta - \delta'); \alpha + \beta + \cdots + \delta + \delta' = n - 2$$

Let next U and V be linearly independent solutions of the equation

$$Y'' + \left(\frac{1-\alpha}{\zeta - a} + \frac{1-\beta}{\zeta - b} + \cdots + \frac{1-\gamma}{\zeta - c}\right) Y' + \frac{\delta\delta'(\zeta - \lambda) \cdots (\zeta - \mu)}{(\zeta - a) \cdots (\zeta - c)} Y = 0 \quad (7.1)$$

In the coefficient of Y, the exponent of the numerator is two units lower than the exponent of the denominator; $\lambda, \cdots \mu$ are "accessory" parameters.

We have then A,B,C,D being constants,

$$w = \frac{F}{Z} = \frac{AU + BV}{CU + DV}, \qquad F = \phi(\zeta)(AU + BV),$$
$$Z = \phi(\zeta)(CU + DV) \quad (7.2)$$

where $\phi(\zeta)$ is a function, which has only regular singularities in the points $\zeta = a,b, \cdots c$, and can be represented in the form

$$\phi(\zeta) = (\zeta - a)^{\alpha'}(\zeta - b)^{\beta'} \cdots (\zeta - c)^{\gamma'} \quad (7.3)$$

The exponents $\alpha', \beta', \cdots, \gamma'$ can be determined in the following manner. Let us write down the conditions for two adjacent sections, for instance the first and the second ones,

$$\operatorname{Im}(k_1Z + l_1F) = \operatorname{Im}(m_1Z + n_1F) = 0, \qquad \operatorname{Im}(k_2Z + l_2F) = \operatorname{Im}(m_2Z + n_2F) = 0$$

and form the equation

$$\begin{vmatrix} k_1\lambda & l_1\lambda & \bar{k}_1 & \bar{l}_1 \\ m_1\lambda & n_1\lambda & \bar{m}_1 & \bar{n}_1 \\ k_2 & 2 & \bar{k}_2 & \bar{l}_2 \\ m_2 & u_2 & \bar{m}_2 & n_2 \end{vmatrix} = 0 \quad (7.4)$$

Designating by λ' and λ'' the roots of this equation, we find

$$\alpha' = \frac{1}{2\pi i}\log \lambda', \qquad \alpha'' = \frac{1}{2\pi i}\log \lambda'' \quad (7.5)$$

Then we must obtain $\alpha' - \alpha'' = \alpha$, where α is equal to the angle of the circular polygon, divided by π. These values α' and α are precisely those which appear in the equations (7.1), (7.2), and (7.3).

α' and α'' are given by the formulas (7.5) within an integer. Eventually they are uniquely determined by inspecting the angles of the figures representing the flow domain, the region of the complex potential, and the velocity hodograph.

The method indicated was given in the papers by Polubarinova-Kochina (123–125,136) and was expanded by Risenkampf (147), Falkovich (31), and Kalinin (57,58).

The method is successful in the case of three singular points, which covers the case of earth dams.

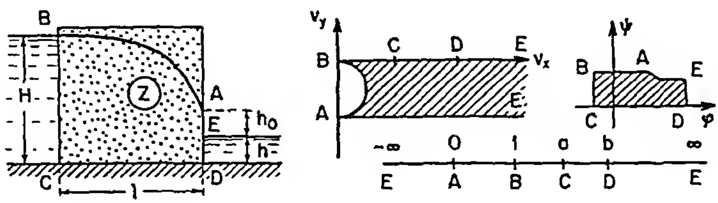


FIG. 11

An *earth dam* with vertical slopes is represented in Fig. 11. In this case it is convenient to make the substitution

$$F = \frac{F_1}{\sqrt{(1-\zeta)(\zeta-a)(b-\zeta)}}, \qquad Z = \frac{Z_1}{\sqrt{(1-\zeta)(\zeta-a)(b-\zeta)}} \tag{7.6}$$

It follows then that for F_1 and Z_1, $\zeta = a$ and $\zeta = b$ are ordinary points, while $0, 1, \infty$ are singular ones. Assume

$$Z_1 = AU + BV, \qquad F_1 = CU + DV \tag{7.7}$$

where U and V are linearly independent solutions of the equation

$$Y'' + \left(\frac{1}{\zeta} + \frac{1}{\zeta - 1}\right) Y' + \frac{1}{4}\frac{Y}{\zeta(\zeta-1)} = 0$$

Assume further

$$K(\zeta) = \int_0^{\frac{1}{2}\pi} \frac{d\varphi}{\sqrt{1 - \zeta \sin^2 \varphi}}$$

Then on the different sections of the real ζ-axis we have

$$Z = \frac{-A[K(\zeta) - iK(1-\zeta)]}{\sqrt{(1-\zeta)(a-\zeta)(b-\zeta)}}, \qquad F = \frac{-\kappa A K(1-\zeta)}{\sqrt{(1-\zeta)(a-\zeta)(b-\zeta)}}$$
$$(0 < \zeta < 1)$$

$$Z = \frac{-AiK(\zeta^{-1})}{\sqrt{\zeta(\zeta-1)(a-\zeta)(b-\zeta)}}, \qquad F = \frac{-\kappa AiK(1-\zeta^{-1})}{\sqrt{\zeta(\zeta-1)(a-\zeta)(b-\zeta)}}$$
$$(1 < \zeta < a)$$

$$Z = \frac{+AK(\zeta^{-1})}{\sqrt{\zeta(\zeta-1)(a-\zeta)(b-\zeta)}}, \qquad F = \frac{\kappa AK(1-\zeta^{-1})}{\sqrt{\zeta(\zeta-1)(\zeta-a)(b-\zeta)}} \qquad (a < \zeta < b) \quad (7.8)$$

$$Z = \frac{+AiK(\zeta^{-1})}{\sqrt{\zeta(\zeta-1)(\zeta-a)(\zeta-b)}}, \qquad F = \frac{\kappa AiK(1-\zeta^{-1})}{\sqrt{\zeta(\zeta-1)(\zeta-a)(\zeta-b)}} \qquad (b < \zeta < \infty)$$

$$Z = \frac{+AK([1-\zeta]^{-1})}{(1-\zeta)\sqrt{(a-\zeta)(b-\zeta)}}, \qquad F = \frac{\kappa AK(\zeta[1-\zeta]^{-1})}{(1-\zeta)\sqrt{(a-\zeta)(b-\zeta)}} \qquad (-\infty < \zeta < 0)$$

Putting $\alpha = 1/a$, $\beta = 1/b$ $(0 \leq \alpha \leq \beta \leq 1)$, and making some trigonometric substitutions, we find for the segments l, H, h, h_0 (Fig. 11), for the discharges Q_h, Qh_0 and $Q = Q_H = Q_h + Q_{h0}$, and for the equation of the free surface the following expressions:

$$x = l - \int_0^{\varphi} \frac{K(\sin^2\varphi)\sin\varphi d\varphi}{\sqrt{(1-\alpha\sin^2\varphi)(1-\beta\sin^2\varphi)}},$$

$$y = \int_0^{\varphi} \frac{K(\cos^2\varphi)\sin\varphi d\varphi}{\sqrt{(1-\alpha\sin^2\varphi)(1-\beta\sin^2\varphi)}}$$

$$l = \int_0^{\frac{1}{2}\pi} \frac{K(\alpha + [\beta-\alpha]\sin^2\varphi)d\varphi}{\sqrt{1-\alpha-(\beta-\alpha)\sin^2\varphi}} = \int_0^{\frac{1}{2}\pi} \frac{K(\sin^2\varphi)\sin\varphi d\varphi}{\sqrt{(1-\alpha\sin^2\varphi)(1-\beta\sin^2\varphi)}}$$

$$H = \int_0^{\frac{1}{2}\pi} \frac{K(\beta + [1-\beta]\sin^2\varphi)d\varphi}{\sqrt{\beta-\alpha+(1-\beta)\sin^2\varphi}} = \frac{1}{\sqrt{\alpha_1}} \int_0^{\frac{1}{2}\pi} \frac{K(1-\beta_1\sin^2\varphi)d\varphi}{\sqrt{1-\gamma_1\sin^2\varphi}} \qquad (7.9)$$

$$h = \sqrt{\gamma} \int_0^{\frac{1}{2}\pi} \frac{K(\alpha\sin^2\varphi)\sin\varphi d\varphi}{\sqrt{(1-\alpha\sin^2\varphi)(1-\gamma\sin^2\varphi)}},$$

$$h_0 = \int_0^{\frac{1}{2}\pi} \frac{K(\cos^2\varphi)\sin\varphi\cos\varphi d\varphi}{\sqrt{(1-\alpha_1\sin^2\varphi)(1-\beta_1\sin^2\alpha)}}$$

$$Q' = \frac{Q}{\kappa} = \frac{1}{\sqrt{\alpha_1}} \int_0^{\frac{1}{2}\pi} \frac{K(\beta_1\sin^2\varphi)d\varphi}{\sqrt{1-\gamma_1\sin^2\varphi}},$$

$$Q_h' = \frac{Q_h}{\kappa} = \sqrt{\gamma} \int_0^{\frac{1}{2}\pi} \frac{K(1-\alpha\sin^2\varphi)\sin\varphi d\varphi}{\sqrt{(1-\alpha\sin^2\varphi)(1-\gamma\sin^2\varphi)}}$$

$$Q_{h_0}' = \frac{Q_{h_0}}{\kappa} = \int_0^{\frac{1}{2}\pi} \frac{K(\sin^2\varphi)\sin\varphi\cos\varphi d\varphi}{\sqrt{(1-\alpha_1\sin^2\varphi)(1-\beta_1\sin^2\varphi)}}$$

$$(\alpha_1 = 1-\alpha, \beta_1 = 1-\beta), \qquad \gamma = \alpha/\beta, \qquad \gamma_1 = \beta_1/\alpha_1$$

In these formulas the notation $A = \frac{1}{2}\sqrt{a^b}$ is used.

By means of the obtained equations, graphs are constructed for discharges and the seepage segment h_0 as a function of the ratios l/H and h/H. For the limiting case of no tail water and an infinitely wide dam ($h = 0$, $l = \infty$), we have $\alpha = 0$, $\beta = 1$, and the equation of the free surface becomes

$$x = \frac{1}{2}\int_0^m \frac{K(m)dm}{1-m}, \qquad y = \frac{1}{2}\int_0^m \frac{K(1-m)dm}{1-m} \qquad (0 < m < 1) \tag{7.10}$$

We have then

$x = 0.0356$	0.0733	0.250	0.490	0.800	1.085	1.520
$y = 0.0838$	0.160	0.445	0.771	0.973	1.224	1.566
$x = 2.35$	5.95	10.87	17.10	24.66	33.55	
$y = 2.13$	3.96	5.77	7.58	9.38	11.19	

Figure 12 represents the corresponding graph. If there is no tail water and $l = \infty$, i.e., in the case of inflow from infinity to the channel,

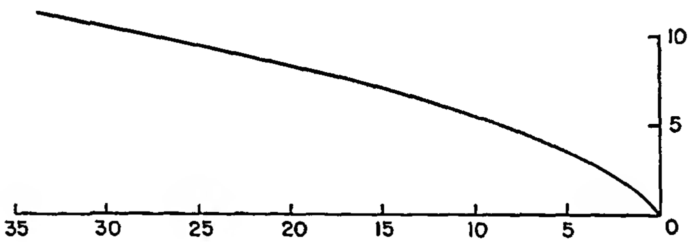


FIG. 12

one gets the simple relation

$$Q = 1.346\kappa h_0$$

Uginchus (164) indicated an approximate method for the calculation of earth dams and checked its accuracy by comparing some of his formulas with the results of Polubarinova-Kochina for a dam. (See also Nelson-Skornyakov (103).)

1. Seepage of Soil Water onto the Downstream Dam Slope

If the upstream dam slope is situated very far from the downstream bank, and there is no tail water, the ground water will leak on the day surface of the downstream slope in the section $BC = l$. Such a problem was investigated by Falkovich (31). Among others, he obtained the relation between l, Q, and the angle $\pi\alpha$ in the following form:

$$l = \frac{4Q}{\kappa\pi^2}\int_0^{\frac{1}{2}\pi} \tan^{1-2\alpha} x \log\cot\frac{x}{2}\,dx \tag{7.11}$$

and plotted the corresponding graph (Fig. 13). It is interesting to note that the function of α is not monotonic.

The following problems can be solved by means of differential equations: (1) those of the filtration with evaporation across a drained dam [Polubarinova-Kochina (127)]; (2) those of the inflow of soil waters to a system of drains located on a waterproof layer, taking account of filtration [Falkovich (31)]; (3) and some problems of filtration in a two-layer medium, treated in section X.

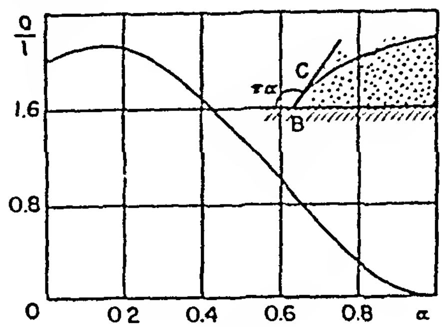


FIG. 13

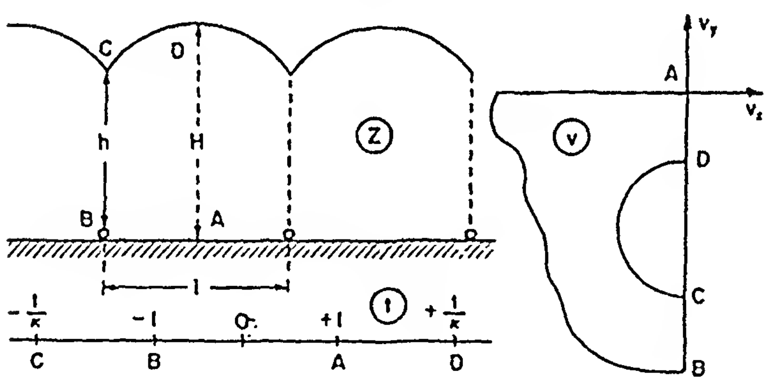


FIG. 14

It is not difficult to write the corresponding linear differential equations for dams with oblique slopes, which is done by Polubarinova-Kochina (128) for the case of no tail water, but, because the equations are complicated, no calculations have been made.

When there are more than three singular points and the lines bounding the region in the hodograph plane do not intersect at one point, the method based on the analytic theory of differential equations reduces to a differential equation which contains unknown accessory parameters. The calculation of these parameters results in complicated transcendental equations; in some special cases it is possible to carry out the calculation. As an example let us consider the inflow of soil waters to a system of drains resting on an impermeable layer [Falkovich (31)].

On a horizontal impermeable layer there is uniformly distributed an infinite series of drain pipes [Fig. 14(z)]. On the horizontal surface there is an inflow of a quantity of water proportional to the horizontal projection of the free surface. The assumed form of the free surface is indicated on Fig. 14(z). Then the velocity hodograph has the form shown in Fig. 14(v). Distributing the singular points as shown in Fig. 14(t), we get the following Riemann scheme

$$u = \frac{v}{(t+1)\sqrt{1-t}}, \qquad v = P\begin{vmatrix} -\frac{1}{k} & -1 & 1 & \frac{1}{k} & \\ 0 & 0 & 0 & 0 & 0t \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \end{vmatrix}$$

Consequently v satisfies the equation

$$\frac{d^2v}{dt^2} + \frac{1}{2}\left(\frac{1}{t+k^{-1}} + \frac{1}{t+1} + \frac{1}{t-1} + \frac{1}{t-k^{-1}}\right)\frac{dv}{dt} + \frac{av}{(t^2-1)(t^2-k^{-2})} = 0$$

where a is the unknown accessory parameter. Putting

$$\tau = \int_0^t \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}} + K, \qquad K = \int_0^1 \frac{dt}{(1-t^2)(1-k^2t^2)}$$

reduces the equation to

$$\frac{d^2v}{dt^2} + ak^2v = 0$$

By integrating this equation, we get

$$\frac{df}{dt} = \frac{A\sin(k\sqrt{a\tau}+\alpha)}{(1+t)\sqrt{1-t}}, \qquad \frac{dz}{dt} = \frac{B\sin(k\sqrt{a\tau}+\beta)}{(1+t)\sqrt{1-t}}$$

Determining from the boundary conditions A,B,k,a, we obtain for the calculation of the accessory parameter a the equation

$$\cos 2k\sqrt{a}\,K = 0$$

VIII. Inverse Methods

Since for given boundaries the determination of the flow of soil waters is frequently very complicated, and for curvilinear boundaries no general methods exist at all, it is sometimes attempted to assume the relations between the unknown functions so as to obtain in the z-plane a contour of desired form.

1. Filtration from Channels

The first problem of this kind was proposed by Hopf and Trefftz (51). It is the problem of the discharge of soil waters intercepted by a "head channel." In 1931 Kozeny (67) gave for it several flow models assuming certain equations between z and f. One of these equations

$$z = -H \exp \frac{-\pi f}{Q} + \frac{if}{\kappa} = -H \exp \frac{-\pi(\varphi + i\psi)}{Q} + \frac{i}{\kappa}(\varphi + i\psi) \quad (8.1)$$

representing the filtration from a channel, has been obtained by Vedernikov (172) in a different way: he assumed in the θ-plane a semicircle which corresponds to the perimeter of the channel. The discharge is expressed by the formula (B is the width of the channel, H is the maximum depth)

$$Q = \kappa(B + 2H) \quad (8.2)$$

Assuming in the equation (8.1) a system of values φ = constant and ψ = constant, Vedernikov gets a flow net, and by means of the formula

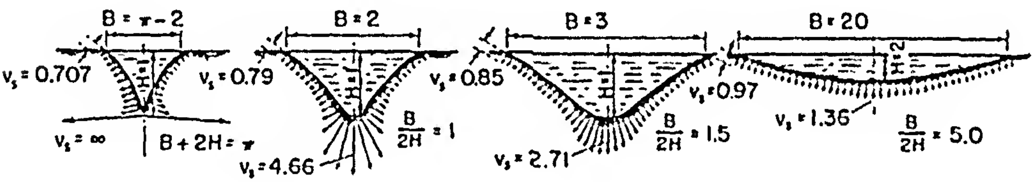


FIG. 15

$p/\gamma = -(\varphi/\kappa + y)$ he plots the isobars. The various forms of the channels and the velocity distribution along their perimeters are represented in Fig. 15, their isobars in Fig. 16.

2. Inflow from Infinity to the Drain Channel

Bazanov (8) assumed that in the plane $dz/df = 1/w$ there corresponds to the channel contour a semicircle, and obtained relatively simple equations. He designed several channels and plotted the flow net for one of them (Fig. 17). Let us remark that in this case the channel boundary is the seepage line, and there is no water in the channel, i.e., a very thin layer only.

3. Earth Dam with a Curvilinear Boundary of the Reservoir

Assuming the contour of the velocity hodograph along the boundary of the head water to be an arc of a circle, designating by α (Fig. 18), the angle between the tangent to the slope line and the x-axis, and assuming

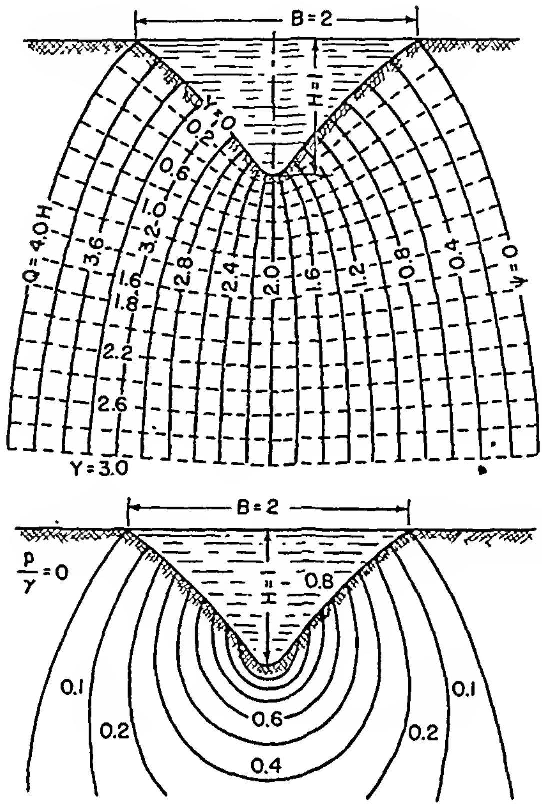


FIG. 16

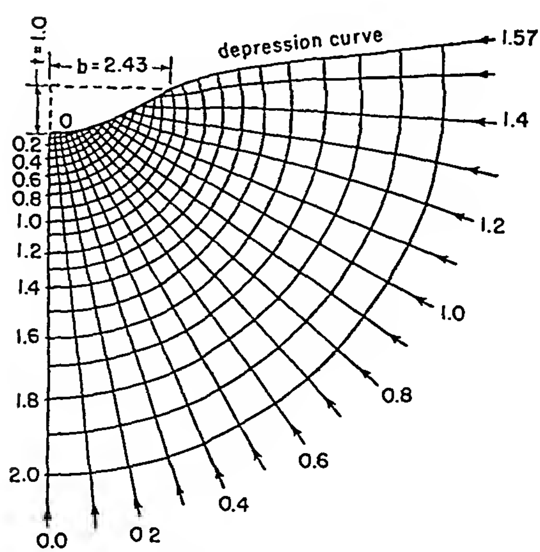


FIG. 17

also $\gamma = (L + l)/H$, Nelson-Skornyakov (91,93) obtained for the flow the following equation:

$$z = \frac{H}{2}\left(\gamma - \frac{1}{2}\tan\alpha\right)\cos\frac{\pi f}{\kappa H} - \frac{f^2}{2\kappa^2 H}\tan\alpha - \frac{if}{\kappa} + \frac{H}{2}\left(\gamma + \frac{1}{2}\tan\alpha\right) \tag{8.3}$$

By separating in the equation for z the real and the imaginary parts, one can plot the flow net. The equation for the upstream slope is obtained

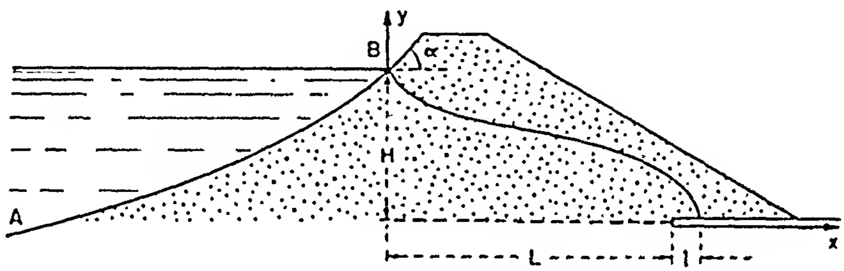


FIG. 18

when $\varphi = -\kappa H$. The discharge Q through the drain segment l is determined from the equation

$$\frac{\pi}{2}\left(\gamma - \frac{1}{2}\tan\alpha\right)\sinh\frac{\pi Q}{\kappa H} + \tan\alpha\,\frac{Q}{\kappa H} + 1 = 0 \tag{8.4}$$

This model has the shortcoming that the line AB descends, though slowly, to infinity. A model free from this shortcoming was proposed by Vedernikov (174,175) in the following form:

4. Dam with a Curvilinear Upstream Slope, but with a Horizontal Bottom of the Upper Reservoir

The author proceeds from the condition that for the Zhukovsky θ-function the contour is a broken line (Fig. 19). The solution is then (β is a parameter)

$$\theta = A\int_0^\zeta \left(\frac{1+\zeta}{\zeta}\right)^\alpha d\zeta \approx A\,\frac{1 + (1+\zeta)^\alpha}{2(1-\alpha)}\,\zeta^{1-\alpha}$$

$$f = \beta\sin^{-1}\sqrt{\frac{\zeta}{\rho}} \tag{8.5}$$

The equation of the free surface is

$$x \approx \frac{H_1}{\pi\alpha(1-\alpha)}\,\zeta^{1-\alpha}\,\frac{1 + (1+\zeta)^\alpha}{2}, \quad \zeta = \beta\sin^2\frac{\pi y}{2H_2} \tag{8.6}$$

In the neighborhood of the downstream slope, this equation reduces to

$$x \approx L + l - \frac{H_1}{\alpha} \beta^{1-\alpha}(1+\beta)^\alpha \cos^2 \frac{\pi y}{2H_2} \tag{8.7}$$

For the width of the drain slit the approximate expression

$$l \approx \frac{\alpha H_2{}^2}{H_1 \beta^{1-\alpha}(1-\beta)^\alpha} \tag{8.8}$$

is obtained.

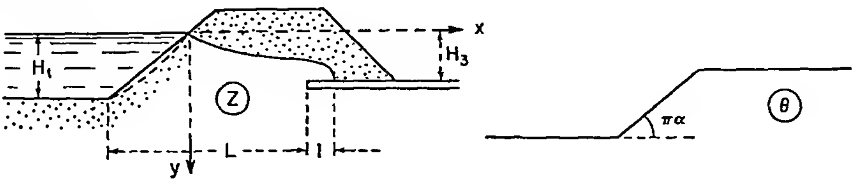


FIG. 19

A pressure slope built on the basis of these formulas differs little from a rectilinear one.

5. Dams with a Nonhorizontal Impermeable Rock

The two dams examined are situated on a permeable base of infinite depth. Voshchinin (188) assumes the permeable base as being of finite

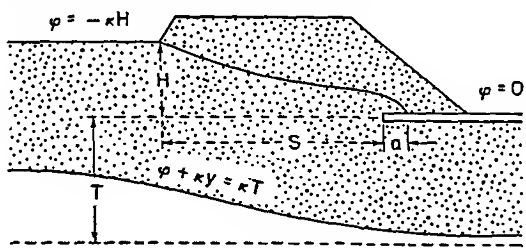


FIG. 20

depth, but its lower boundary becomes curvilinear, since he takes as condition on the lower boundary (besides ψ = constant) the constancy of Re $\theta = \varphi + \kappa y$. He neglects the influence of the pressure slope (Fig. 20).

The relation between z and f becomes

$$\tanh \frac{\pi}{2T}\left(\frac{fi}{\kappa} + z - \frac{S}{2}\right) = k sn\left(\frac{2Kf}{\kappa H} + K, k\right) \quad \left(k = \tanh \frac{\pi S}{4T}\right) \tag{8.9}$$

where k is the modulus of the elliptic function sn, and K the full elliptic integral with the modulus k.

For the discharge Voshchinin obtains

$$Q = \frac{\kappa H K'}{2K}$$

The lower boundary of the permeable layer is a curve which in infinity deviates from the horizontal by the distance $\pm\frac{1}{2}H$.

To find the drain segment a it is necessary to substitute in the equation

$$\tanh \frac{\pi}{2T}\left(\frac{S}{2} - \psi - a\right) = -\frac{k}{dn(2K\psi/H,k')} \quad (k' = \sqrt{1 - k^2}) \tag{8.10}$$

the value

$$\psi_0 = \frac{H}{2K}\int_0^\delta \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}}\left(\delta = \frac{1}{\sqrt{1+b^2}},\, b = \frac{2Ktk}{\pi H}\right) \tag{8.11}$$

Nelson-Skornyakov (98) considers a different model of a drained dam. The layer of water in the head water is also assumed as infinitely thin. At a certain depth under the dam the author places a vertical impermeable semi-infinite segment; he gets then a family of parabola-shaped streamlines, which bypass this segment, and whose branches go down into infinity. Any of these streamlines can be taken as a rigid boundary. Nelson-Skornyakov's (99) problem, mentioned at the end of section III, can be considered also to be an inverse problem. Nelson-Skornyakov (99) in his book solves a series of "limiting" problems, i.e., problems with relatively simple solutions, defining boundaries such that the real cases are bracketed by the "limiting" ones. However, it is not always possible to assert that the flow elements in the intermediate cases will have values intermediate between those obtained in the limiting cases. This is, for instance, shown by the graph in Fig. 13.

IX. Other Methods

1. *Reduction to Hilbert's Problem*

Numerov developed a method for problems in which the domain of the complex potential is known. Following the author, we denote it by

$$\omega = \varphi + i\psi,\ d\omega/dz = v_x - iv_y \tag{9.1}$$

The coordinate z is considered a function of ω.

On different segments of the contour of the ω region the author obtains for the unknown analytic function $Z(\omega) = X + iY$ the condition

$$aX + bY = c \tag{9.2}$$

where a, b are constants, and c the given function of the boundary arc. This is the Riemann-Hilbert problem.

As an example, Numerov (110) considers filtration through an earth

dam with trapezoidal cross section, located on an impermeable base, with drain in the downstream part, without an interval of seepage (Fig. 21).

Let us write the relation between the velocity potential and the pressure in the form [see (1.4)]

$$\varphi = \operatorname{Re} \omega = \frac{\kappa(h_1 + h_2)}{2} + \frac{\kappa p_0}{\rho g} - \kappa y - \frac{\kappa p}{\rho g} \tag{9.3}$$

and let us introduce an auxiliary function

$$Z(\omega) = \frac{1}{2kQ}\left[\frac{\kappa(h_1 + h_2)}{2} - \omega\right]^2 + z(\omega) \tag{9.4}$$

For this function we get the following equations of the type (9.2):

On the line M_1M_2, where $y = 0$, $\psi = 0$

$$Y = 0$$

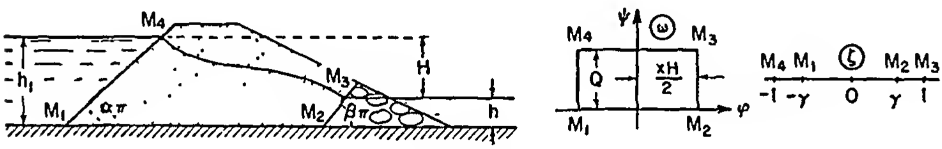


FIG. 21

On M_2M_3, where $\varphi = \frac{1}{2}\kappa H$

$$X \sin \pi\beta - Y \cos \pi\beta = \left(l + \frac{\kappa h_2}{2Q}\right)^2 \sin \pi\beta + \frac{h_2 \cos \pi\beta}{Q}\psi - \frac{\sin \pi\beta}{2\kappa Q}\psi^2$$

On M_3M_4, where $p = p_0$, $\psi = Q$, $\varphi = \frac{1}{2}\kappa(h_1 + h_2) - \kappa y$

$$Y = 0$$

On M_1M_4

$$X \sin \pi\alpha - Y \cos \pi\alpha = \frac{\kappa h_1{}^2 \sin \pi\alpha}{2Q} + \frac{h_1 \cos \pi\alpha}{Q}\psi - \frac{\sin \pi\alpha}{2\kappa Q}\psi^2$$

Then the author maps the rectangle of the ω region onto the semiplane ζ (Fig. 21),

$$\zeta = k\operatorname{sn}\left(\frac{2K\omega}{\kappa H}, k\right) \tag{9.5}$$

k being determined from the equation

$$\frac{Q}{\kappa H} = \frac{K'}{2K} \tag{9.6}$$

The author then introduces a new function

$$W(\zeta) = iA\zeta + iB\zeta^2 + w(\zeta)Z(\zeta)$$

where

$$w(\zeta) = i(k - \zeta)^{\beta}(k + \zeta)^{1-\alpha}(1 - \zeta)^{1-\beta}(1 + \zeta)^{\alpha}$$

On the segments of the real axis of the ζ-plane he gets more convenient conditions for $W(\zeta)$, W being an analytic function in the upper semi-plane, including the point $\zeta = \infty$, single valued and continuous up to the real axis, on which the following conditions hold:

$$f(\xi) = \operatorname{Re} W = \begin{cases} \left[+b_0 - \frac{b_1}{K} F(\lambda,k') - \frac{b_2}{K'^2} F^2(\lambda,k')\right] \phi_1(\xi)(k \leq \xi \leq 1) \\ \left[-a_0 + \frac{a_1}{K'} F(\lambda,k') + \frac{a_2}{K'^2} F^2(\lambda,k')\right] \phi_2(\xi)(-1 \leq \xi \leq \gamma) \\ (0\ |\xi| \leq \gamma, |\xi| \geq 1) \end{cases} \tag{9.7}$$

where

$$\lambda = \text{arc sin} \sqrt{\frac{1 - \xi^2}{k'}} \qquad \phi_1(\xi) = (\xi - k)^{\beta}(k + \xi)^{1-\alpha}(1 - \xi)^{1-\beta}(1 + \xi)^{\alpha}$$

$$\phi_2(\xi) = (k - \xi)^{\beta}(-k - \xi)^{1-\alpha}(1 - \xi)^{1-\beta}(1 + \xi)^{\alpha}$$

$$a_0 = h_1 \cos \alpha\pi + \frac{\kappa h_1^2 \sin \alpha\pi}{2Q} - \frac{Q \sin \alpha\pi}{2\kappa}$$

$$b_0 = L \sin \beta\pi + h_2 \cos \beta\pi - \frac{\kappa h_2^2 \sin \beta\pi}{2Q} - \frac{Q \sin \beta\pi}{2\kappa}$$

$$a_1 = h_1 \cos \alpha\pi - \frac{Q \sin \alpha\pi}{\kappa}, \qquad b_1 = h_2 \cos \beta\pi - \frac{Q \sin \beta\pi}{\kappa}$$

$$a_2 = \frac{Q \sin \alpha\pi}{2\kappa}, \qquad b_2 = \frac{Q \sin \beta\pi}{2\kappa}$$

Now, instead of linear combinations of two functions X and Y, only one function is given: the real part of W. This is the Dirichlet problem for a semi-plane, solved by means of the Cauchy type integral

$$W(\zeta) = -\frac{i}{\pi}\int_{-1}^{+1} \frac{f(t)dt}{t - \zeta} + iC + iA\zeta + iB\zeta^2 + w(\zeta)Z(\zeta)$$

Substituting in this equality successively $\zeta = \pm k$, $\zeta = \pm 1$ and subtracting, one eliminates the constants A,B,C and gets $Z(\zeta)$ in the form

$$Z(\zeta) = -\frac{1}{\pi}(k - \zeta)^{1-\beta}(k + \zeta)^{\alpha}(1 - \zeta)^{\beta}(1 + \zeta)^{1-\alpha}$$

$$\int_{-1}^{+1} \frac{f(t)dt}{(k^2 - t)(1 - t^2)(t - \zeta)} \int_{-1}^{+1} \frac{f(t)dt}{(k^2 - t^2)(1 - t^2)} = 0 \tag{9.8}$$

Finally, for $z(\omega)$ the solution becomes

$$z(\omega) = -\frac{1}{2\kappa Q}\left[\frac{\kappa(h_1+h_2)}{2} - \omega\right]^2$$
$$-\frac{1}{\pi}(k-\zeta)^{1-\beta}(k+\zeta)^{\alpha}(1-\zeta)^{\beta}(1+\zeta)^{1-\alpha}\int_{-1}^{+1}\frac{f(t)dt}{(k^2-t^2)(1-t^2)(t-\zeta)}$$

ζ being determined by (9.5) and f by the formulas (9.7). Q and k are calculated by means of (9.6) and (9.8).

If $L/H > 10$, then $Q/\kappa H$ is small and k near unity, and $k' \approx 0$ and $F(t,k') \approx t$. With these assumptions the equation (9.8) yields

$$L \approx \frac{\kappa(h_1^2 - h_2^2)}{2Q} + h_1[\cot \alpha\pi - f_1(\alpha)\cos \alpha\pi] - h_2 f_1(\beta)\cos \beta\pi$$
$$- \frac{Q}{\kappa}\left[\frac{1}{2} - f_2(\alpha)\sin \alpha\pi - f_3(\beta)\sin \beta\pi\right]$$

where

$$f_1(\alpha) = \int_0^1 t\cot^{1-2\alpha}\frac{\pi t}{2}\,dt, \qquad f_2(\alpha) = \int_0^1 t\left(1-\frac{t}{2}\right)\cot^{1-2\alpha}\frac{\pi t}{2}\,dt,$$
$$f_3(\alpha) = \frac{1}{2}\int_0^1 t^2\cot^{1-2\alpha}\frac{\pi t}{2}\,dt$$

Assuming $\omega = \varphi + iQ$, and assuming that at $k \approx 1$, $sn(u,k) \approx thu$, the author obtains for the depression curve the following equations:

For the left-hand half of the curve:

$$x \approx \frac{\kappa(h_1^2 - y^2)}{2Q} + h_1\left[\cot \pi\alpha - F_1\left(\tanh\frac{\pi\kappa(h_1 - y)}{2Q}, \alpha\right)\right]$$
$$+ \frac{Q}{\kappa}\tan \pi\alpha F_2\left(\tanh\frac{\pi\kappa(h_1-y)}{2Q}, \alpha\right) \quad (9.9)$$

For the right-hand half of the curve

$$x \approx L + \frac{\kappa(h_2^2 - y^2)}{2Q} + h_2 F_3\left(th\frac{\pi\kappa(h_2+y)}{2Q}, \beta\right)$$
$$+ \frac{Q}{\kappa}F_4\left(th\frac{\pi\kappa(h_2+y)}{2Q}, \beta\right) \quad (9.10)$$

The author gives tables for the functions $F_i(u,\alpha)(i = 1, \cdots, 4)$ for both arguments.

For a dam with a vertical upstream slope the equation of the depression curve is given by the author in the form

$$x = \frac{\kappa}{Q} y \left(H - \frac{y}{2}\right) + \frac{Q}{\kappa} F \left(th \frac{\pi \kappa y}{2Q}\right) \tag{9.11}$$

Instead of the four functions F_i, there remains only one:

$$F(u) = \frac{2}{\pi} \int_0^1 \tan^{-1}\left(u \tan \frac{\pi t}{2}\right) \left[1 - \frac{1}{\pi} \tan^{-1}\left(u \tan \frac{\pi t}{2}\right)\right] dt \tag{9.12}$$

At the end of the paper the author investigates the condition for the absence of the seepage interval.

In the paper by Numerov (104) there is given the calculation of two dams, one of rectangular, the other of a trapezoidal cross section. The same paper has a section on the application of electrohydrodynamic analogies for plotting flow nets. In other papers (105,108) there are discussed problems of bank drains running parallel to the banks of a water reservoir or channel, on an impermeable base of infinite (105) or finite depth, in the presence of evaporation or infiltration on parts of the free surface. In one paper (106) Numerov is concerned with the problem on filtration through an earth dam with an inclined upstream slope and a horizontal drainage on an infinitely deep permeable base. In two papers (109,111) he gives tables of auxiliary functions which appear in the formulas derived by the author.

2. *Applications of Functional Analysis*

These applications are made by Gersevanov. Designating the velocity potential and the stream function (appropriately normed) respectively by α and β

$$\omega - \varphi + i\psi = \alpha + i\beta, \qquad d\omega/dz = -v_x + iv_y \tag{9.13}$$

the author writes the relation between the complex coordinate of the flow and the complex potential in the form $x + iy = f(\alpha + i\beta)$; hence

$$x = \frac{1}{2}[f(\alpha + i\beta) + \bar{f}(\alpha - i\beta)], \qquad y = \frac{1}{2i}[f(\alpha + i\beta) - \bar{f}(\alpha - i\beta)]$$

Using part of the boundary conditions of the one or the other problem, the author writes for the function f a functional equation of one of the types

$$f(x + p) \pm f(x) = rx + s \quad \text{or} \quad f(px) = f(x) \tag{9.14}$$

p,r,s being constants.

The author seeks the solution of these equations in form of series of exponentials. By this method he gets solutions of special problems concerning some cases of filtration from channels and the flow around a

cutoff in a permeable layer of finite depth. For a cutoff in an infinite region (semi-plane), the author obtains an infinite number of solutions which satisfy the boundary conditions; but the solution for which the condition at infinity is prescribed (in this problem the velocity is to be zero, in other cases the velocity may be a constant or have a certain order of growth) is the only one which coincides with the one already known.

Regarding the problem on the earth dam without a seepage interval, we make the following remark. Filtration in a dam is impossible without a seepage segment, unless the flow satisfies some additional conditions. In Gersevanov's solution, as shown by the study of the velocities, there is on the boundary of the tail water a point vortex which causes the lowering of the depression curve.

3. Variational Methods

In his work in jet theory, Lavrentiev obtained some theorems pertaining to the following problem: How is the mapping function and its derivative affected by small changes of the boundary of a given region? These results he also applied to the motion of soil waters (76).

Assume an apron $(0,l)$ with cutoffs whose lengths are $l_1, l_2 \cdots, l_p$ (Fig. 1), the depth of the impermeable layer being h. Variational theorems lead to a number of qualitative conclusions concerning the changes in the basic flow elements [such as discharge, exit velocity (along the boundary of the tail water), and pressure on the apron] in response to changes in the dimensions of the elements of the installation.

If the length of one or several cutoffs is increased, all the flow lines are lowered, and the discharge and the exit velocity are reduced. The most effective means of reducing the exit velocity is to lengthen the extreme right cutoff.

If the length of a cutoff is increased, the pressure on the apron is increased to the left of this cutoff, and reduced to the right of it; in particular, an increase of the length of the extreme right apron increases the pressure on the apron everywhere.

The same theorems permit justifying and making more accurate the approximate methods of solution of problems on filtration, and estimating the errors of these methods. Lavrentiev analyzes from this viewpoint the "fragment method" of Pavlovsky which replaces the actual equipotential lines $\gamma_1', \gamma_2' \cdots$ (Fig. 1) by vertical segments $\gamma_1'', \gamma_2'' \cdots$. The author points out the possibility of increasing the accuracy of the method of fragments by taking as γ_i'' half of a sine wave. The author indicates then a procedure for the modification of the velocity potential when modifying slightly a structure, and this specifies the procedure for the case when the length of one cutoff is changed.

X. Filtration in Multilayer Soils

Let us assume that the flux of soil waters filtered through a soil of filtration coefficient κ_1 crosses some surface S into another soil of a filtration coefficient κ_2; the velocity vector suffers then a sudden change across such a surface of separation. Designating by $\mathbf{v}_1$ and $\mathbf{v}_2$, the filtration velocities on the surface S in the first and second soil, we have the following boundary conditions:

$$v_{1n} = v_{2n}, \qquad \frac{v_{1t}}{k_1} = \frac{v_{2t}}{k_2} \tag{10.1}$$

The first condition is the continuity equation, and the second is obtained from the Darcy law and the continuity of the pressure across the separation surface S. Thus the plane problem reduces to the determination of two functions of a complex variable $df_1/dz = v_{1x} - iv_{1y}$ and $df_2/dz = v_{2x} - iv_{2y}$, which determine the flow in the first and in the second soil, and which satisfy the conditions (10.1) on the separation line.

In the general form the solution of this problem can be reduced to a system of singular integral equations, but this method is of little use for numerical calculations or the derivation of analytical relations.

Particular problems have been solved by several methods, such as (1) construction of the flow from the singularities, (2) application of the Fourier integral, (3) application of the analytic theory of differential equations.

1. Construction of the Flow from Singularities

As an example let us consider the flow represented on Fig. 22. The liquid which filters from the water reservoir of depth H_1 into the soil with the filtration coefficient κ_1 and thickness h goes in part into the water reservoir of depth H_2, and in part into the soil of different filtration coefficient κ_2, whereupon it enters into the water reservoir of depth H_3.

To find the complex velocities $v_{1x} - iv_{1y}$ and $v_{2x} - iv_{2y}$ which determine the flow in the first and second soils, we note that for $y = 0$ both functions are imaginary, and for $y = -h$ real. Hence both functions can be analytically continued: the first on the entire semi-plane $x < 0$, the second on the semi-plane $x > 0$. Moreover, the function $v_{1x} - iv_{1y}$ has simple poles at the points $z = -a \pm 3nih$ ($n = 0, \pm 1, \pm 2, \cdots$) with the residues $(-1)^n\kappa_1(H_1 - H_2)i/\pi$; similarly the function $v_{2x} - iv_{2y}$ is a function determined in the semi-plane $x \geq 0$ with simple poles in the points $z = b \pm 2nih$ and residues equal to $(-1)^n\kappa_2(H_2 - H_3)i/\pi$.

To satisfy the boundary conditions (10.1) on the separation line, let us place at the points $z = a \pm 2nhi$ and $z = -b \pm 2nhi$ vortices with

intensities $(-1)^n\kappa_1(H_1 - H_2)\lambda i/\pi$ and $(-1)^n\kappa_2(H_2 - H_3)(1 - \lambda)i/\pi$. Here and in the sequel $\lambda = (\kappa_2 + \kappa_1)\ (\kappa_2 - \kappa_1)$.

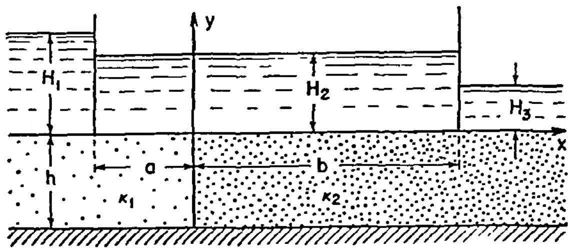


FIG. 22

We get then for the unknown functions the following expressions

$$v_{1x} - iv_{1y} = \frac{\kappa_1 i}{h}\left[(H_1 - H_2)\operatorname{cosech}\frac{\pi(z+a)}{h}\right.$$
$$\left. + \lambda(H_1 - H_2)\operatorname{cosech}\frac{\pi(z-a)}{h} + (1+\lambda)(H_2 - H_1)\operatorname{cosech}\frac{\pi(z-b)}{h}\right] \tag{10.2}$$
$$v_{2x} - iv_{2y} = \frac{\kappa_2 i}{h}\left[(1-\lambda)(H_1 - H_2)\operatorname{cosech}\frac{\pi(z+a)}{h}\right.$$
$$\left. + (H_2 - H_3)\operatorname{cosech}\frac{\pi(z-b)}{h} - \lambda(H_2 - H_3)\operatorname{cosech}\frac{\pi(z+b)}{h}\right]$$

By the same method solutions are obtained for a system of drain pipes and for two symmetrically situated dams (Fig. 23). The last problem

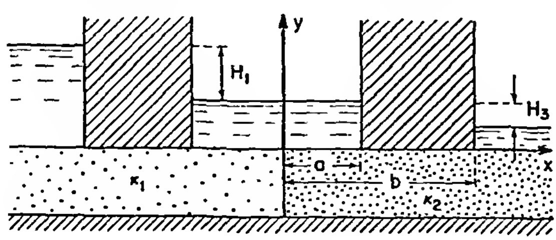


FIG. 23

leads to the construction of a purely circular flow around a double net. The solution has the form

$$v_{1x} - iv_{1y} = \left(A\cosh\frac{2\pi z}{h} + B\sinh\frac{2\pi z}{h}\right)\Big/ \sqrt{\left(\cosh^2\frac{2\pi a}{h} - \cosh^2\frac{2\pi z}{h}\right)\left(\cosh^2\frac{2\pi b}{h} - \cosh^2\frac{2\pi z}{h}\right)}$$

$$v_{2x} - iv_{2y} = \left(\cosh\frac{2\pi z}{h} + D\sinh\frac{2\pi z}{h}\right)\Big/ \sqrt{\left(\cosh^2\frac{2\pi a}{h} - \cosh^2\frac{2\pi z}{h}\right)\left(\cosh^2\frac{2\pi b}{h} - \cosh^2\frac{2\pi z}{h}\right)}$$

$$A = \frac{\kappa_1(1+\lambda)(H_1+H_2)}{2J_1}i, \qquad B = \frac{\kappa_1[H_2(1+\lambda) - H_1(1-\lambda)]}{2J_2}$$

$$C = \frac{\kappa_2(1-\lambda)(H_1-H_2)}{2J_1}, \qquad D = \frac{\kappa_2[H_1(1-\lambda) - H_2(1+\lambda)]}{2J_2} \tag{10.3}$$

$$J_1 = \frac{h}{2\pi}\operatorname{cosech}\frac{2\pi b}{h}\int_0^{\frac{1}{2}\pi}\frac{d\varphi}{\sqrt{1-p^2\sin^2\varphi}},$$

$$p^2 = 1 - \sinh^2\frac{2\pi a}{h}\operatorname{cosech}^2\frac{2\pi b}{h}$$

$$J_2 = \frac{h}{2\pi}\operatorname{sech}\frac{2\pi b}{h}\int_0^{\frac{1}{2}\pi}\frac{d\varphi}{\sqrt{1-q^2\sin^2\varphi}},$$

$$q^2 = 1 - \cosh^2\frac{2\pi a}{h}\operatorname{sech}^2\frac{2\pi b}{h}$$

The solution of the last two problems by the method of differential equations is given by Polubarinova-Kochina (137). The construction of the solution from singularities belongs to Falkovich.

2. Construction of the Flow by Means of the Fourier Integral

The representation of the solution of multilayer problems in the form of the Fourier integral is possible in the case when there are only point singularities, and when the separation lines of the layers with different filtration coefficients are parallel straight lines or straight lines intersecting at one point. This method was proposed by Risenkampf (148).

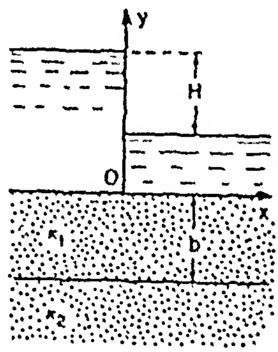


FIG. 24

Let us consider the problem, solved by Risenkampf, and concerning the filtration of water, under a thin wall separating two water reservoirs, into soil consisting of two layers with filtration coefficients κ_1 and κ_2; the second layer extends indefinitely downwards (Fig. 24). We seek the complex velocity in the first and second layers in the form:

$$\begin{aligned} v_{1x} - iv_{1y} &= \frac{\kappa_1 Hi}{\pi}\left[\frac{1}{z} + i\int_{-\infty}^{+\infty} a(u)e^{-iuz}du\right] \\ v_{2x} - iv_{2y} &= -\frac{\kappa_2 H}{\pi}\int_{-\infty}^{+\infty} b(u)e^{-iuz}du \end{aligned} \tag{10.4}$$

where $a(u)$ and $b(u)$ are unknown functions of the real variable u.

Representing the function $1/z$ in the region $0 > y \geq -b$ in the form

$$\frac{1}{z} = i \int_0^\infty e^{-iuz} du$$

and satisfying the boundary conditions on the separation line of the soil, we determine $a(u)$ and $b(u)$. The solution has the form

$$\begin{aligned} v_{1x} - iv_{1y} &= \frac{\kappa_1 H i}{\pi} \left[\frac{1}{z} + 2\lambda \int_0^\infty \frac{\sin uz du}{e^{2bu} - \lambda} \right] \\ v_{2x} - iv_{2y} &= \frac{\kappa_2 H i (1 - \lambda)}{\pi} \int_0^\infty \frac{e^{-iuz} du}{1 - \lambda e^{-u}} \end{aligned} \tag{10.5}$$

Polubarinova-Kochina (136) showed that by the method of the Fourier integral, one can solve the problem for an arbitrary number of layers of arbitrary thicknesses and different filtration coefficients. Distributing the singularities along some segment of the curve, one can obtain from the solution of the problem discussed an integral equation which determines the solution of the problem of filtration under the apron [Polubarinova-Kochina (137), Kalinin (55), Risenkampf (148)]. Lavrentiev and Pogrebissky (77) distribute the vortices along the separation line of the soils. One can hereby assume an arbitrary number of drain pipes in the soil. [Topolyansky (162,163)] Falkovich (31) solved the problem, represented in Fig. 24, when the separation line of the soils is inclined to the horizon. In this case it is necessary to use the Riemann-Mellin integral instead of the Fourier integral.

3. *Three-Dimensional Sink in a Two-Layer Medium*

Millionshchikov (87) solved with the aid of the Fourier integral the following problem. At the point $z = a$, between the planes $z = 0$ and $z = h$, there is a sink. The plane $z = 0$ is impermeable so that in it the vertical velocity is equal to zero. The region $z > h$ is filled with a medium of another permeability. Putting

$$\varphi_i = -c_i \left(\frac{p_i}{\rho g} + z \right) \left(c_i = \frac{\kappa_i}{\mu_i} \right)$$

and assuming $i = 0$ for the upper region, and $i = 1$ for the lower one, we find, from the continuity of the pressure and the normal velocity in the separation plane of the two media, that for $z = h$

$$c_0 \varphi_0 = c_1 \varphi_1, \qquad \frac{\partial \varphi_0}{\partial z} = \frac{\partial \varphi_1}{\partial z}$$

The solution φ_1 can be represented in a series form $(r = \sqrt{x^2 + y^2})$

$$\varphi_1 = -\frac{Q_1}{4\pi}\left\{\frac{1}{[r^2 + (a - z)^2]^{1/2}} + \sum_{n=1}^{\infty} \frac{\lambda^n}{[r^2 + (2nh - a + z)^2]^{1/2}} + \sum_{n=1}^{\infty} \frac{\lambda^n}{[r^2 + (2nh + a - z)^2]^{1/2}} + \sum_{n=0}^{\infty} \frac{\lambda^n}{[r^2 + (2nh + a + z)^2]^{1/2}} + \sum_{n=1}^{\infty} \frac{\lambda^n}{[r^2 + (2nh - a - z)^2]^{1/2}}\right\} \quad \left(\lambda = \frac{c_1 - c_0}{c_1 + c_0}\right) \tag{10.6}$$

The pattern of the flow is such, as if at the points $\pm 2nh \pm a$ there existed sinks with intensities decreasing in a geometric progression.

If $c = 0$, i.e., the upper layer is impermeable, we get $\lambda = 1$ and the series for ϕ becomes divergent; in this case one can use for φ the formula given in the book on Bessel functions by Gray, Mathews, and McRoberts or in Bateman's book. (z' is the sink point, $z = \pm c$ are the rigid walls.)

$$\varphi = \int_0^\infty \gamma(z,z')J_0(lr)dl, \qquad \gamma(z,z') = \begin{cases} \dfrac{2\cosh l(c - z)\cosh l(z' + c)}{\sinh 2lc}, & z' \leq z \leq c \\[2ex] \dfrac{2\cosh l(c - z')\cosh l(c + z)}{\sinh 2lc}, & -c \leq z < z \end{cases} \tag{10.7}$$

4. Method of Differential Equations

Polubarinova-Kochina (126,137) showed that the method of differential equations can also be used for the solution of problems of filtration in a two-layer soil if the regions occupied by the two soils are the same and bounded by segments of straight lines. As an example of the application of this method let us consider the filtration in a two-layer soil, consisting of two horizontal layers of filtration coefficients κ_1 and κ_2 (Fig. 25), and of equal thickness h. The upper layer has an apron. The lower layer is situated on an impermeable base. Putting

$$\zeta = \frac{1 + a}{2} + \frac{1 - a}{2}\cosh\frac{\pi z}{h} \quad \left(a = \tanh^2\frac{\pi l}{2h}\right) \tag{10.8}$$

let us map the region A_1DCBA goes into the upper semi-plane, and the region $ABEA_2$ into the lower.

Investigating the functions $F_1 = df_1/d\zeta$, $F_2 = df_2/d\zeta$, where $f_k = \varphi_k + i\psi_k$, we get for them conditions shown on the corresponding segments

(Fig. 25). Using these conditions, one can continue the function F_1, defined in the upper semi-plane, into the lower semi-plane, and the function F_2 from the lower into the upper semi-plane, by considering F_1 and F_2 as linearly independent integrals of a linear differential equation of second order.

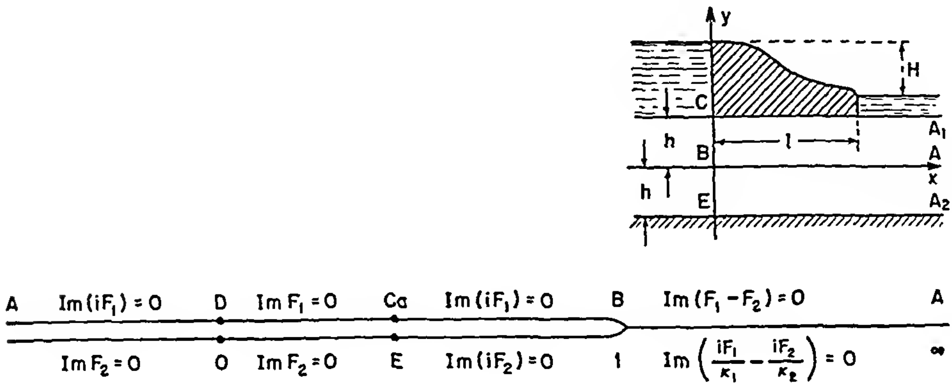


FIG. 25

Analyzing the boundary conditions on the real axis, and putting $F_1 = \phi_1/\sqrt{\zeta - a}$, $F_2 = \phi_2/\sqrt{\zeta - a}$, we find that the functions ϕ_1 and ϕ_2 represent branches of the Riemann function

$$P\begin{vmatrix} 0 & \infty & 1 & \\ 0 & 1+\epsilon & 0 & \zeta \\ -\frac{1}{2} & 1+\epsilon & -\frac{1}{2} & \end{vmatrix} \qquad \left(\tan \pi\epsilon = \sqrt{\frac{\kappa_2}{\kappa_1}}\right)$$

Finally we get

$$F_1 = \frac{Bi\{(\sqrt{1-\zeta} + \sqrt{-\zeta})^{2\epsilon} - (\sqrt{1-\zeta} - \sqrt{-\zeta})^{2\epsilon}}{2\sqrt{\zeta(1-\zeta)(\zeta - a)}}$$

$$F_2 = -\frac{B\{(\sqrt{1-\zeta} + \sqrt{-\zeta})^{2\epsilon} - (\sqrt{1-\zeta} - \sqrt{-\zeta})^{2\epsilon}\}}{2\sqrt{\zeta(1-\zeta)(\zeta - a)}} \tan \pi\epsilon$$

$$B = \frac{\kappa_1 H}{2J}, \; J = \int_0^{\sin^{-1}\kappa} \frac{\cos 2\epsilon\alpha\, d\alpha}{\sqrt{k^2 - \sin^2 \alpha}} \qquad \left(k = \tanh \frac{\pi l}{2h}\right)$$

For the complex velocities in the first and second medium we have

$$v_{1x} - iv_{1y} = \frac{\kappa_1 H\pi}{4hJ}$$

$$\frac{(-\sin wiz + i\sqrt{\cosh^2 wl - \sin^2 wiz})^{2\epsilon} + (-\sin wiz - i\sqrt{\cosh^2 wl - \sin^2 wiz})^{2\epsilon}}{\sqrt{\cosh^2 wl - \sin^2 wiz}}$$

$$v_{2x} - iv_{2y} = \frac{\kappa_1 H \pi i}{4hJ}$$
$$\frac{(\sin wiz + i\sqrt{\cosh^2 wl - \sin^2 wiz})^{2\epsilon} - (\sin wiz - i\sqrt{\cosh^2 wl - \sin^2 wiz})^{2\epsilon}}{\sqrt{\cosh^2 wl - \sin^2 wiz}} \tag{10.9}$$

where $\omega = \pi/2h$.

In the paper by Polubarinova-Kochina there are also given formulas for the velocity along the boundaries, and a graph is plotted for (the relation between) $Q/\kappa_1 H$ against ϵ and l/h.

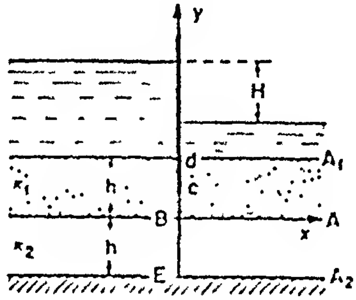


FIG. 26

By the same method Polubarinova-Kochina solved the problem of filtration in a two-layer soil in which a cutoff is driven to a depth d (Fig. 26). A special case of this problem, when $d = h$, was investigated earlier by Girinsky (41); Kalinin (56) obtained the solution of the problem of filtration under an apron, shown in Fig. 27. The latter problem reduces to a differential equation with five singular points, and the solution is obtained in terms of the theta functions.

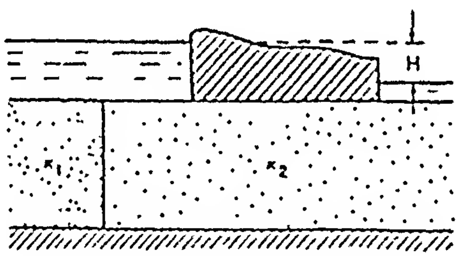


FIG. 27

5. *Hydraulic Theory of a Well in a Two-Layer Medium*

Myatiev (89) investigates the problem on filtration in a permeable layer taking into account the low permeability of the neighboring layers. Designating by z the pressure, taken as the average along the height of the layer, by H the value of the pressure at infinity, and setting

$$H - z = S$$

he gets the differential equation

$$\frac{d^2S}{dr^2} + \frac{1}{r}\frac{dS}{dr} - \lambda S = 0, \quad \left[\lambda = \left(\kappa_2 m_3 \frac{m_1}{\kappa_1} + \frac{m_3}{\kappa_3}\right)^{-1}\right] \tag{10.10}$$

where m_1 is the depth of the upper layer of filtration coefficient κ_1, m_3 is the depth of the following layer of low permeability and filtration coefficient κ_3, and m_2 is the depth of the middle, main layer (of filtration coefficient κ_2), for which S is wanted. Some generalization of this theory is given by Polubarinova-Kochina (144), who presents the following derivation of Myatiev's equation.

Let us average the continuity equation (1.3) along the thickness m_n of the layer.

$$\frac{\partial}{\partial x}\int_0^{m_n} v_x dz + \frac{\partial}{\partial y}\int_0^{m_n} v_y dz + \int_0^{m_n} \frac{\partial v_z}{\partial z} dz = 0$$

or

$$\frac{\partial}{\partial x}(m_n V_x) + \frac{\partial}{\partial y}(m_n V_y) + (v_z)m_n - (v_z)_0 = 0 \tag{10.11}$$

Here V_x, V_y are the average velocities; let us introduce the average value of the pressure

$$H_n(x,y) = \frac{1}{m_n}\int_0^{m_n} h_n(x,y,z)dz; \qquad V_x = -\kappa_n \frac{\partial H_n}{\partial x}; \qquad V_y = -\kappa_n \frac{\partial H_n}{\partial y}$$

For a layer of constant thickness, assuming that the vertical velocities on the top and bottom of the layer result from seepage through the neighboring upper and lower layers of low permeability, of thicknesses m_{n-1} and m_{n+1}, of filtration coefficients κ_{n-1} and κ_{n+1}, and average pressures H_{n-1} and H_{n+1}, we have

$$(v_z)m_n = \kappa_{n-1}\frac{H_n - H_{n-1}}{m_{n-1}}, \qquad (v_z)_0 = \kappa_{n+1}\frac{H_{n+1} - H_n}{m_{n+1}}$$

$$m_n\kappa_n\left(\frac{\partial^2 H_n}{\partial x^2} + \frac{\partial^2 H_n}{\partial y^2}\right) - \frac{\kappa_{n-1}}{m_{n-1}}(H_n - H_{n-1}) + \frac{\kappa_{n+1}}{m_{n+1}}(H_{n+1} - H_n) = 0 \tag{10.12}$$

which for $n = 2$ gives (10.10) and

$$H = \left(\frac{\kappa_{n+1}}{m_{n+1}}H_{n+1} + \frac{\kappa_{n-1}}{m_{n-1}}H_{n-1}\right)\Big/\left(\frac{\kappa_{n+1}}{m_{n+1}} + \frac{\kappa_{n-1}}{m_{n-1}}\right)$$

If we have a flow with a free surface, m_n is a variable; one can take $m_n = H_n$, and we get Myatiev's generalization of Dupuit's equation

$$\kappa_n\left[\frac{\partial}{\partial x}\left(H_n\frac{\partial H_n}{\partial x}\right) + \frac{\partial}{\partial y}\left(H_n\frac{\partial H_n}{\partial y}\right)\right] - w + \frac{\kappa_{n+1}}{m_{n+1}}(H_{n+1} - H_n) = 0 \tag{10.13}$$

where w characterizes the infiltration or evaporation with a free surface.

If $\kappa_{n+1} = \kappa_{n-1} = 0$ is assumed, (10.12) gives us the Laplace equation, and (10.13) the Dupuit equation (the subscript n is omitted)

$$\frac{\partial^2 H^2}{\partial x^2} + \frac{\partial^2 H^2}{\partial y^2} = w \tag{10.14}$$

For a well of radius δ situated along the z-axis, the solution (10.10) takes the form

$$S = H - z = \frac{E}{2\pi\rho\kappa_3 m_3 \sqrt{\lambda}} \frac{K_0(r\sqrt{\lambda})}{K_1(\delta\sqrt{\lambda})} \tag{10.15}$$

where E is the output of the well, and K_0 and K_1 are Bessel's functions of second order and imaginary argument.

Calculation with the last formula gives good agreement with experiment.

XI. Filtration of Oil

Following Dupuit (30), Forchheimer (33), and Zhukovsky (195), it is usual to assume in the basic problems of oil filtration in a porous medium (sand) that a slightly curved layer is horizontal and limited by two impermeable horizontal planes. If the well is perfect, i.e., passes through the entire thickness of the oil layer, one may consider the flow as plane. The well is assumed to be a source. If at the points $z_1, \cdots,$ z_n n wells of the intensities $q_1, \cdots, q_n$ are situated, the complex potential of the flow has the form

$$f(z) = -\frac{q_1}{2\pi}\log(z - z_1) - \cdots - \frac{q_n}{2\pi}\log(z - z_n) + C$$

The diameters of the wells are small in comparison with the distances between the wells; hence the lines of equal potential, near to the points z_k differ little from circles and may be taken as the contours of the cross sections of the wells.

The output of a well is determined by the Dupuit formula

$$q = \frac{2\pi bk}{\mu} \frac{p_k - p_c}{\log(R/\delta)} \tag{11.1}$$

where p_k is the pressure at the distance R from the well center, p_c the pressure on the well contour, and δ the well radius.

In the problem of rational distribution of wells it is important to find the relation between the output of all the wells on the one hand, and the number of wells, the distance between them, their relative position, the value of the pressures in the wells, etc., on the other. Such problems are

contained in Leibenson's book (78); the investigations are continued and developed in detail in the paper by Shchelkachev and Pykhachev (156).

The concept of the "influence radius" of wells was introduced in the first papers on the inflow of soil waters and of oil to wells. This concept is subjected in the book of the aforementioned authors to a critical analysis, and replaced by the concept of the feed region, i.e., a region in which water reserves are accumulated. They consider the so-called water pressure condition at which the filtration phenomenon may be schematized as follows. There exists a region occupied by a porous medium; the boundaries of the region are surfaces of equal pressure (equal potential) and rigid walls (faults). In a plane motion the feeding contour is an isobar (line of equal pressure).

If the feed contour is a circumference of radius R and center at the origin of coordinates, and if the well is at the point $Z_1 = X_1 + Y_1 i$, the complex potential is

$$f(z) = -\frac{q_1}{2\pi}\left[\log (Z - Z_1) - \log \frac{R^2 - \bar{Z}Z_1}{R}\right] + C$$

$$\varphi(x,y) = -\frac{q}{2\pi}\left[\log |Z - Z_1| - \log \frac{|R^2 - Z\bar{Z}_1|}{R}\right] + C_1 \qquad (11.2)$$

If the pressure on the circumference is equal to p_k and on the well contour, which is a circle of radius δ equal to p_c, the output of the well is given by the formula

$$q = \frac{2\pi\kappa(p_k - p_c)}{\mu[\log (R^2 - X_1{}^2 - Y_1{}^2) - \log (R\delta)]} \qquad (11.3)$$

If the feed contour C has an arbitrary form, we map the region bounded by it, with the help of the equation $Z = F(z)$, onto the interior of the circle of radius R, and find that the output of the well at the point z_0 is

$$q_1 = \frac{2\pi\kappa(p_k - p_c)}{\mu\{\log (R^2 - |F(z_0)|^2) - \log [R\delta/|F'(z_0)|]\}} \qquad (11.4)$$

In the paper by Polubarinova-Kochina (139) there are given graphs (Fig. 28) in which the output q_1 of the well at the center of an elliptical region is plotted against the output q_0 of the well in a circular region of radius R. The curve I corresponds to the case when the ellipses are equal to the circle $R^2 = ab$. The curve II is obtained for ellipses having the same small semi-axis $b = R$. It is seen that the output changes very little with the change of the form of the feed region. In many problems it is possible, therefore, to use the simplest regions only: a circle, a strip, a semi-plane, etc. [See Charny (13,14).]

It would be interesting to study the *inverse problems of the filtration theory*, i.e., problems of finding from given well outputs quantities such as the permeability of the layer, the dimensions of the region, etc. Polubarinova-Kochina (138) investigated the following cases: (1) to find a circular form of the region occupied by the filtering liquid, given the outputs of three or more wells turned on one after another; (2) the region is assumed to be an infinite strip; (3) the oil deposit of circular form is surrounded by water.

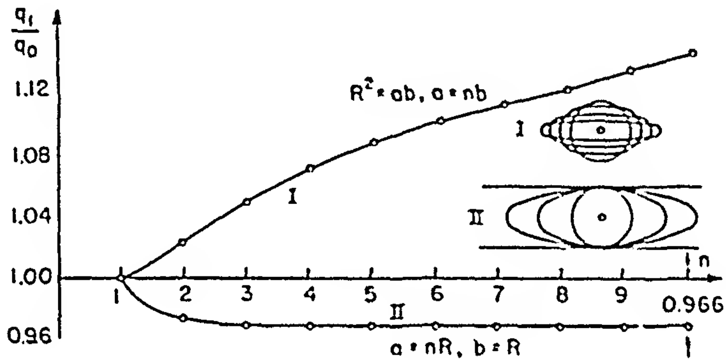


FIG. 28

As an example, we show how to find the radius R and the center coordinates (a,b) of a circular region, if n wells with the coordinates x_i, $y_i(i = 1, 2, \cdots n)$ are turned on one after another and their outputs

$$q_i = \frac{2\pi\kappa(p_k - p_i)}{\mu} \bigg/ \log \frac{R^2 - (x_i - a)^2 - (y_i - b)^2}{\delta R} \tag{11.5}$$

are measured, p_k being the pressure on the feed contour and p_i the pressure on the well. Writing this equation in the form

$$(x_i - a)^2 + (y_i - b)^2 = R(R - r_i) \left(r_i = \delta \exp \frac{2\pi\kappa(p_k - p_i)}{\mu q_i} \right), \qquad (i = 1, 2, \cdots, n)$$

and subtracting one of each pair of such equations from the other, one obtains for the determination of $a,b,$ and R a system of equations:

$$\begin{aligned} &2a(x_i - x_j) + 2b(y_i - y_j) - R(r_i - r_j) = \rho_{ij} \\ &x_i^2 + y_i^2 - x_j^2 - y_j^2 = \rho_{ij} \end{aligned} \qquad (i, j = 1, 2, \cdots, n)$$

which may be solved for $a,b,$ and R by the method of least squares.

Charny (16) treats the problem on *the most favorable distribution of wells in oil layers under water pressure conditions.* The author approaches

the solution of the problem "hydraulically," i.e., he takes the average velocity on the layer cross section, the entire layer being considered as a tube (Fig. 29). It is assumed that the area of the cross section of the layer is a function of the length s. On the feed contour $p = p_k$, on the contour of the "gallery" (line which replaces a group of wells, and for which $s = l$) $p = p_c$. One gets then, with the help of the Darcy law, for the propagation velocity of the separation boundary ds/dt

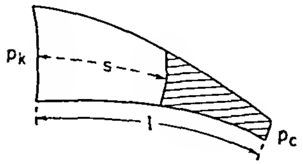


FIG. 29

$$q = fm\frac{ds}{dt} = (p_k - p_c)\left[\mu_B \int_0^s \frac{ds}{\kappa f} + \mu_H \int_s^1 \frac{ds}{\kappa f}\right]^{-1} \tag{11.6}$$

where q is the volume discharge in the pipe, m is the porosity; the subscript B refers to water, H to petroleum. Hence

$$t = \frac{m}{\Delta p}\int_{s_0}^{s} f\left(\mu_B \int_0 \frac{ds}{\kappa f} + \mu_H \int_s^l \frac{ds}{\kappa f}\right) ds \tag{11.7}$$

The problem is to extract the petroleum in minimum time by means of n galleries turned on one after another, as follows: when the water reaches the first gallery (the rest of them being turned off), this gallery is turned off, and the following one is turned on; and so forth. Then the total working time is given by the formula

$$t_{\text{on}} = \frac{m}{\Delta p}\sum_{i=1}^{n}\int_{s_{i-1}}^{s_i} f\left(\mu_B \int_0^s \frac{ds}{\kappa f} + \mu_H \int_s^{s_i} \frac{dt}{\kappa f}\right) ds \tag{11.8}$$

To find the s_i at which t_{on} is a minimum, Charny differentiates t_{on} with respect to s_i, sets the derivatives equal to zero, and gets the equations

$$\frac{1}{(\kappa f)_i}\int_{s_{i-1}}^{s_i} f ds - f_i \int_{s_i}^{s_{i+1}} \frac{ds}{\kappa f} = 0$$

In the case of a strip-like deposit, one has f = constant and

$$s_i - s_{i-1} = s_{i+1} - s_i = \text{constant} \tag{11.9}$$

i.e., the distances between the series of the wells must be equal.

If $f = as$, one gets

$$\log\frac{s_i}{s_{i+1}} = \frac{s_{i-1}^2 - s_i^2}{2s_i^2}$$

This case occurs for radial flow. In the two latter cases the solution of the problem is exact.

1. Inflow of Liquid to Wells in a Heterogeneous Medium

The problem of the influence of the heterogeneous soil composition upon the well output is of interest in the theory of petroleum filtration. More important is the evaluation of the influence of the difference of viscosities of water and oil upon the well output. But the inflow to the well of oil surrounded by water presents a complicated problem in unsteady flow in which the interface between water and oil changes with time. However, if one considers small time intervals during which the interface does not change appreciably, one may use the results for steady flow, assuming that we have two regions with equal soil permeability filled with liquids of different viscosities.

Let us assume that in the first and second region we have respectively

$$\varphi_1 = -c_1 p_1, \quad \varphi_2 = -c_2 p_2 \quad (c_i = \kappa_i/\mu_i)$$

Consider a well in a semi-plane, in a region of permeability κ_1; or, else, consider the upper semi-plane filled with oil of viscosity μ_1; in the lower semi-plane we have a soil of permeability κ_2 or, correspondingly, water of viscosity μ_2. The pressure p_c on the well boundary and the pressure p_k on some isobar very remote from the well are known.

The complex flow potentials in the regions 1 and 2, as shown by Polubarinova-Kochina (138), are then

$$f_1 = \varphi_1 + i\psi_1 = -\frac{Q}{2\pi}[\log(z - z_0) + \lambda \log(z - \bar{z}_0)] + \text{constant}$$

$$\left(\lambda = \frac{c_1 - c_2}{c_1 + c_2}\right) \tag{11.10}$$

$$f_2 = \varphi_2 + i\psi_2 = -\frac{Q}{2\pi}(1 - \lambda)\log(z - z_0) + \text{constant}$$

For the well output the formula

$$Q = \frac{2\pi c_1(p_k - p_c)}{\log(R/\delta) + \lambda \log(R/2b)} \tag{11.11}$$

is derived, where δ is the well radius, R the distance from the well to the intersection point of the interface with the isobar for the pressure p_k (the feed contour).

In the region 1 the isobars are "generalized lemniscates" $|z - z_0| |z - \bar{z}_0|^\lambda =$ constant; in the region 2 they are circles.

Consider next a well in the case when the regions are limited by concentric circles with radii $r = 1$ and $R = 1$. In the interior region the complex velocity is w_1 (the well is at the point $z = d$), in the ring region it is w_2.

The expressions for them are

$$u_1 - iv_1 = -\frac{Q}{2\pi}\left\{\frac{1}{z-\alpha} + \frac{1}{z-\alpha^{-1}} + (1-\lambda^2)\sum_{n=0}^{\infty}\frac{z^n\bar{z}_0^{\,n+1}}{R^{2n+2}-\lambda}\right\}$$

$$u_2 - iv_2 = -\frac{Q}{2\pi}\left\{\frac{1-\lambda}{z-\alpha} + \frac{\lambda}{z} + (1-\lambda)\sum_{n=0}^{\infty}\frac{z^n\bar{z}_0^{\,n+1}}{R^{2n+2}-\lambda} + \lambda(1-\lambda)\sum_{n=1}^{\infty}\frac{z_0^n}{z^{n+1}(R^{2n}-\lambda)}\right\} \tag{11.12}$$

When the outer region is infinite, i.e., $R \to \infty$, in each of the right-hand members only two terms remain. The obtained expressions may be given the form

$$u_1 - iv_1 = -\frac{Q}{2\pi}\left(\frac{1}{z-\alpha} + \frac{\lambda}{z-\bar{\alpha}^{-1}} - \sum_{n=1}^{\infty}\frac{\lambda^{n-1}}{z-R^{2n}/\bar{\alpha}}\right) \tag{11.13}$$

$$u_2 - iv_2 = -\frac{Q(1-\lambda)}{2\pi}\left(\frac{1}{z-\alpha} + \sum_{n=1}^{\infty}\frac{\lambda^n}{z-\alpha/R^{2n}} - \sum_{n=1}^{\infty}\frac{\lambda^{n-1}}{z-R^{2n}/\bar{\alpha}}\right)$$

It is seen that by summation one may obtain the solution for an arbitrary number of wells.

Pykhachev (121) gives an approximate solution for a ring battery of n wells, situated concentrically with the circle $|z| = 1$.

XII. Three-Dimensional Problems

The first example of a *possible three-dimensional flow* has been indicated by Risenkampf and Kalinin (149). They noticed that in the flow around an ellipsoid, the velocity potential on the ellipsoidal surface is a linear function of the coordinates, while in the problem of soil waters, on the surface, $\varphi = -\kappa y +$ constant, as we know. By a suitable selection of constants, one may make the given surface of the ellipsoid serve as a free surface for the flow of the soil waters.

Ivakin (52) assumes in his paper that *the form of the free surface, in the case of pumping through the well bottom* into homogeneous unsaturated soil, is nearly ellipsoid of revolution. The author schematizes the flow by introducing a source into the advancing flow (Prandtl's half-body).

Filtration of oil in nonhorizontal layers. When studying filtration of oil, it is sometimes necessary to consider the distortion of the oil layer. When the oil deposit has the form of a cupola or of a cylindrical dome, one gets relatively simple formulas for the inflow of the liquid to a well,

orthogonal to the boundaries (spheres or cylinders) of the deposit. Such a problem is investigated by Kazarnovskaya and Polubarinova-Kochina (123), and Zarevich. The character of the displacement of the water-oil interface (assuming their viscosities and densities equal) was investigated for a spherical cupola. The interface was initially assumed to be horizontal.

Kazarnovskaya (62) investigated in detail the displacement of a water-oil interface in an oblique layer limited by two parallel planes containing an infinite series of wells under the same assumptions as in the preceding problem). She found simple formulas for the area of the watered part of the well at two different times (this area is proportional to the water discharge into the well). She also obtained the equation of the water-oil interface at any moment, if it was initially a horizontal plane. The author gives also a formula which permits us approximately to take into account the presence of several parallel series of wells.

Imperfect well. An important three-dimensional problem of another kind is the problem of an imperfect well, i.e., one not reaching the bottom of the layer. Muskat gave an approximate solution of this problem, in which he substituted the well by a system of sinks whose intensities are chosen so that on the cylindrical surface of the well, the normal velocity component is equal to zero. Millionshchikov (87) imposes another condition: $v_\rho = 0$ along the cylindrical surface of the well, i.e., for $\rho = \delta$ (radius of the well) and for $0 < z < H - \epsilon$; but $v\rho = -Q/2\pi\delta\epsilon$ is given on the segment $H - \epsilon < z < H$.

We get, then, for the velocity potential:

$$\varphi = -\frac{Q}{\pi^2\delta}\sum_{n=1}^{\infty}\frac{1}{n}\frac{K_0(n\pi\rho/H)}{K_0(n\pi\delta/H)}\cos\frac{n\pi(H-\epsilon)}{H}\cos\frac{n\pi z}{H} + \frac{Q}{2n}\log\rho \quad (12.1)$$

For $\epsilon \to 0$, $\cos[n\pi(H - \epsilon)/H] \to (-1)^n$, we get the potential for an infinitely narrow slit on the well. Millionshchikov uses it for approximate determination of the interface in the case of a slit-shaped well, and of oil flowing over resting water. The author determines the form of the interface in such a steady motion from the continuity of pressure across this surface. This leads to the equation (the subscript B refers to water, H to oil, and γ is the specific gravity)

$$\varphi_H(\rho,z) + \frac{\kappa_H}{\mu_H}(\gamma_B - \gamma_H)z = 0$$

Assuming the equation

$$z = \frac{\mu_H}{\kappa_H(\gamma_B - \gamma_H)}\varphi(\rho,z)_{z=0}$$

as the equation of the interface, the author gets the relation in dimensionless coordinates (Fig. 30).

System of imperfect wells. Segal (154) investigated a three-dimensional grid of imperfect wells; in other words, an imperfect well in a parallelepiped, the upper and lower bases of which are rigid walls (the

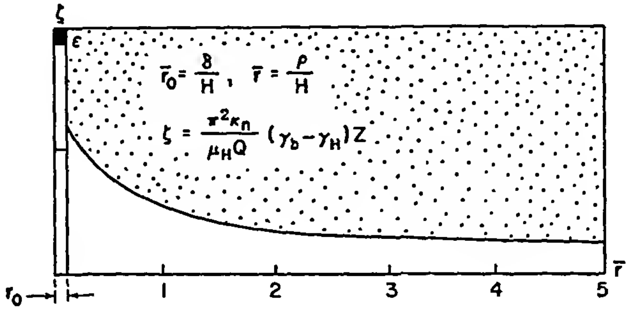


Fig. 30

normal velocity component is equal to zero), and on the vertical boundaries of which either φ = constant or $\partial\varphi/\partial n = 0$.

Two basic problems are illustrated in Fig. 31, and they represent in cross section: (1) a series of equidistant wells, equally distant from the rectilinear feed contours, (2) a series of equidistant wells with the feed contour and rift (fault trace; rigid boundary) parallel to the axis of this series.

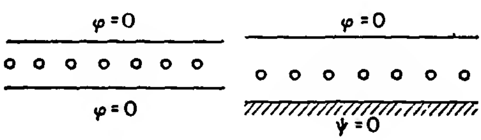


Fig. 31

To solve this problem Segal considers first the velocity potential of a group of N sources and sinks of intensities q_n, located in the basic parallelepiped with the edges α, β, γ in the points (ξ_n,η_n,ζ_n). This group of sources and sinks is repeated in parallelepipeds filling the entire space, whose vertices have the coordinates $l_1\alpha$, $l_2\beta$, $l_3\gamma$, where l_1, l_2, l_3 are integers varying from $-\infty$ to $+\infty$. The velocity potential of such a three-dimensional grid is expressed by the series

$$\varphi(x,y,z) = \frac{1}{4\pi} \sum_{l_1=-\infty}^{+\infty} \sum_{l_2=-\infty}^{+\infty} \sum_{l_3=-\infty}^{+\infty} \sum_{n=1}^{N} \frac{q_n}{R_{ln}}$$

$$R_{ln} = \sqrt{(l_1\alpha + \zeta_n - x)^2 + (l_2\beta + \eta_n - y)^2 + (l_3 + \zeta_n - z)^2}$$

which converges only conditionally. Therefore he considers first the convergent series

$$\varphi_\lambda(x,y,z) = \frac{1}{4\pi} \sum_l \sum_{n=1}^{N} \frac{q_n}{R_{l_n}} \exp\left(-\lambda R_{l_n}\right)$$

After transforming this series into a form convenient for calculations, the passage to the limit $\lambda \to 0$ is made; one gets for φ the expression

$$\varphi(x,y,z) = \frac{1}{4\pi^2\alpha\beta\gamma} \sum_l \frac{\sigma_l}{hl} \exp\left(\frac{-\pi^2 hl}{T^2} - 2\pi i kl\right)$$

$$+ \frac{1}{4\pi} \sum_l \sum_{n=1}^{N} q_n \frac{1 - erf(TR_{l_n})}{R_{l_n}}$$

Here is

$$erf(u) = \frac{2}{\sqrt{\pi}} \int_0^u e^{-t^2} dt, \qquad \sigma l = \sum_{n=1}^{N} q_n \exp\left(2\pi i k_{l_n}\right)$$

$$k_{l_n} = \frac{\xi_n l_1}{\alpha} + \frac{\eta_n l_2}{\beta} + \frac{\zeta_n l_3}{\gamma}, \qquad hl = \frac{l_1^2}{\alpha^2} + \frac{l_2^2}{\beta^2} + \frac{l_3^2}{\gamma^2}$$

$$kl = \frac{x l_1}{\alpha} + \frac{y l_2}{\beta} + \frac{z l_3}{\gamma}$$

T being a positive number chosen for fast convergence of both series in the final formula for φ. In each special problem the expression for φ is simplified further.

Then Segal distributes along the well axis a line of sinks whose intensities he assumes as given by a second-degree polynomial

$$q(\zeta) = A + B\zeta + C\zeta^2, \quad (0 \leq \zeta \leq h)$$

where h is the well depth. At the end of the wells a point sink of intensity D is assumed. The constants A,B,C,D are selected to satisfy the condition

$$\frac{\partial \phi}{\partial n} = 0 \left(\phi = \int_0^h q(\zeta)\varphi(x,y,\zeta)d\zeta\right)$$

on the lateral surface of the well cylinder.

The well output is calculated from the formula

$$Q = Ah + B\frac{h^2}{2} + C\frac{h^3}{3} + D$$

The author also supplies some numerical examples.

Charny (17) gives an approximate method for the output computation in the same cases as those investigated by Segal. Assuming that at a certain distance from the imperfect well the flow may be considered plane, Charny separates a cylindrical surface which may be considered an equipotential one and reduces the problem of a system of wells to a problem of one well, for the output of which Muskat (88) gave an approximate formula.

XIII. Unsteady Flows

1. Flow under Dams at Variable Pressures

Davison made the first study of unsteady flows with filtration under hydraulic structures when the pressures in the head and tail filtration under hydraulic structures and the pressures in the head and tail waters are given functions of time. Davison takes the equations of unsteady motion in the form

$$\frac{1}{m}\frac{\partial v_x}{\partial t} = -\frac{1}{\rho}\frac{\partial p}{\partial x} - \frac{g}{\kappa}v_x, \quad \frac{1}{m}\frac{\partial v_z}{\partial t} = \frac{1}{\rho}\frac{\partial p}{\partial z} - \frac{g}{\kappa}v_z, \tag{13.1}$$

$$\frac{1}{m}\frac{\partial v_y}{\partial t} = -\frac{1}{\rho}\frac{\partial p}{\partial y} - \frac{g}{\kappa}v_y - g, \quad \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} = 0$$

Introducing the vector

$$\mathbf{V} = v - \frac{\kappa}{m}\frac{\partial v}{\partial t} \tag{13.2}$$

Davison arrives at the equation

$$\mathbf{V} = -\kappa \operatorname{grad}\left(\frac{p}{\rho g} + y\right) \tag{13.3}$$

i.e., the equation of steady flow.

The problem of finding a velocity potential which on the boundary of the kth water reservoir takes the value

$$\varphi = -\kappa H_k(t) \quad (k = 1, 2, \cdots, n) \tag{13.4}$$

with $\partial\varphi/\partial n = 0$ on the impermeable boundaries, reduces to the solution of n problems of the following kind: to find the solution of Laplace's equation $\Delta\varphi_k = 0$ for which $\partial\varphi_k/\partial n = 0$ on the impermeable boundaries, $\varphi_k = 1$ on the boundary of the kth water reservoir and $\varphi_k = 0$ on the remaining boundaries of the water reservoirs. Then the expression

$$\varphi(x,y,z,t) = -\kappa[H_1(t)\varphi_1(x,y,z) + \cdots + H_n(t)\varphi_n(x,y,z)] \tag{13.5}$$

gives the desired potential.

If at the time $t = 0$ the velocity vector has the value $\mathbf{v}_0(x,y,z)$, then integrating the equation (13.2), in which we replace $\mathbf{v}$ by grad φ, where φ

is taken from (13.5), we get

$$v = v_0 - \exp\frac{-gmt}{\kappa} - mg\sum_{i=1}^{n}\operatorname{grad}\varphi_i(x,y,z)\exp\frac{-gmt}{\kappa}\int_0^t \exp\frac{gmt_i}{\kappa}H_i(t_i)dt_i \tag{13.6}$$

The exponential function exp $(-mgt/\kappa)$ damps out quickly with the time, which supports the opinion of some authors, that the derivatives with respect to time in the equations (13.1) should not be taken into account at all. In this case the solution of the problem would be given by the equation $v = -\kappa \operatorname{grad}\varphi$, where φ is determined by the formula (13.5).

As concrete examples Davison considers the flow around one or two aprons with variable working pressures.

Another kind of problem of unsteady flow is investigated by Polubarinova-Kochina (137). It is the problem on the propagation of the interface between two liquids of different densities (fresh and salt waters) under hydraulic construction. Here the problem reduces to the solution of the telegraph equation.

2. *The Problem of Contraction of an Oil-Bearing Region*

There exists a series of papers on theory of filtration dealing with the problem first proposed by Leibenson (78), of the contraction of an oil-bearing contour. Polubarinova-Kochina (140,141), Galin (36), Kalinin (59). Kufarev and Vinogradov (74) worked on this problem. In the papers of the latter two authors there is given a rigorous justification of the methods applied by the former authors, i.e., developments in series, and for two particular cases a solution in a simple and closed form is obtained.

The problem is stated as follows. In a porous medium there is initially given a plane region, bounded by the contour C_1 filled with petroleum and containing several wells (point sinks). On the contour C the pressure is assumed constant at all times. It is asked how the contour C and the outputs of the wells will change in time.

In the actual problem the region occupied by the oil is surrounded by a region occupied by water so that we have to consider the flow of two liquids of different densities and viscosities. The continuity of pressure in the outer region is equivalent to the assumption that in the outer region the liquid has a viscosity equal to zero. Galin performs a conformal mapping of the flow region z onto the circle $|\zeta| < 1$, the well, which is at the point z_0, going into the center of the circle $\zeta = 0$. Let the mapping function be

$$z = f(\zeta,\tau) \quad (f(0,\tau) = z_0) \tag{13.7}$$

where the variable τ is connected with the time t by the relation

$$t = -\int^{\tau} q(\tau)d\tau \tag{13.8}$$

$2\pi q(\tau)$ being the intensity of the sink $(q(\tau) < 0)$. We seek a function holomorphic in ζ $f(\zeta,t)$, one-leafed for $|\zeta| < 1$, and satisfying the boundary condition

$$\operatorname{Re}\left[\frac{1}{\zeta}\frac{\partial \bar{z}}{\partial \zeta}\frac{\partial z}{\partial \tau}\right]_{|\zeta|=1} = 1 \tag{13.9}$$

Example 1. The initial region is a semi-plane Re $z < 0$. Let $z_0 = 1$. Then

$$f_0(\zeta) = f(\zeta,0) = \frac{\zeta - 1}{\zeta + 1}$$

If the solution is sought in the form of a series

$$f(\zeta,t) = \sum_{n=1}^{\infty} f_n(\zeta)\tau^n \tag{13.10}$$

recurrent boundary equations

$$\operatorname{Re}\left[\frac{f_1(\zeta)}{\zeta f_0'(\zeta)}\right] = |f_0'(\zeta)|^2$$

$$\operatorname{Re}\left[\frac{f_n(\zeta)}{\zeta f_0'(\zeta)}\right] = -\frac{1}{n|f_0'(\zeta)|^2}\sum_{k=1}^{n-1} k \operatorname{Re}\left[\frac{1}{\zeta} f_k(\zeta)\bar{f}_{n-k}(\zeta^{-1})\right] \tag{13.11}$$

for $f_n(\zeta)$ are obtained.

Kufarev and Vinogradov (74) succeeded in finding $f_n(\zeta)$ and summing the series for z, and obtained as a result

$$z = f(\zeta,\tau) = \frac{\zeta - 1}{\zeta + 1} + \frac{\zeta(\zeta + 3)}{3(\zeta + 1)}(-1 + \sqrt{1 + 6\tau}) \tag{13.12}$$

This function is the solution of the problem for $-\frac{1}{6} < \tau < \infty$.

Example 2. The initial region is a strip $|x| < h$. The center of the well is at the point $z = 0$. In this case

$$f_0(\zeta) = \frac{2h}{\pi i}\log\frac{1 + i\zeta}{1 - i\zeta}$$

$$z = f(\zeta,\tau) = \frac{2h}{\pi i}\log\frac{1 + i\zeta}{1 - i\zeta} + \frac{\zeta}{\pi}(-2h + \sqrt{4h^2 + 2\pi^2\tau})$$

$$\left(-\frac{2h^2}{\pi^2} < \tau < \infty\right) \tag{13.13}$$

Polubarinova-Kochina (140–142) gives the conditions on the contour in another form. Differentiating (with respect to time) the equation $\varphi(x,y,t)$ = constant, which must be satisfied for all t, she gets the condition (taking into account that $\partial x/\partial t = m^{-1}\partial\varphi/\partial x, \cdots$)

$$\left(\frac{\partial\varphi}{\partial x}\right)^2 + \left(\frac{\partial\varphi}{\partial y}\right)^2 + m\frac{\partial\varphi}{\partial t} = 0 \tag{13.14}$$

which is then reduced to several equations depending upon the case considered: one or several wells within a circular layer, one or a chain of wells in the semi-plane. She and Kalinin found for the examples shown only the first terms of the power series in t or ζ, because the evaluation of the following terms is cumbersome. A closed simple solution is obtained when the initial contour is a cardioid:

$$z = \zeta + a\zeta^2 \quad (\zeta = e^{i\varphi})$$

where $a < \frac{1}{2}$ (otherwise the cardioid has a double point and the mapping is not one-to-one), the well being at the point $z = 0$. The solution has the form

$$z = A_1(t)\zeta + A_2(t)\zeta^2$$

where $A_1(t)$ and $A_2(t)$ are solutions of the system

$$A_1\dot{A}_1 + 2A_2\dot{A}_2 = \frac{\log\delta}{\log(A_1/\delta)}, \quad A_1\dot{A}_2 + 2A_2\dot{A}_1 = 0$$

δ being the radius of the well, and

$$\dot{A}_1 = \frac{dA}{dt}, \quad A_1(0) = 1, \quad A_2(0) = a$$

The integration of the system yields

$$t = 1 + 2a^2 - \frac{1-a^2}{2\log\delta} + \frac{1}{\log\delta}\left(A_1^2 + \frac{2a^2}{A_1^4}\right)\log\frac{A_1}{\delta} - \frac{1}{2\log\delta}\left(A_1^2 - \frac{a^2}{A_1^2}\right), \quad A_2 = \frac{a}{A_1^2}$$

3. *Problem of the Depression of the Soil-Water Level under the Action of a Drain Pipe* (59)

In the problems of the inflow of liquid to drain pipes, when there is a free surface, the condition p = constant on the free surface is rigorously satisfied. On account of (1.4) this condition is equivalent to $\varphi + \kappa y$ = constant.

Differentiating with respect to t, we get

$$\left(\frac{\partial\varphi}{\partial x}\right)^2 + \left(\frac{\partial\varphi}{\partial y}\right)^2 + \kappa\frac{\partial\varphi}{\partial y} + m\frac{\partial\varphi}{\partial t} = 0 \tag{13.15}$$

Mapping the flow region onto the semi-plane ζ, we see that the unknown functions $z(\zeta,t)$ and $\theta = f - i\kappa z$ satisfy on the real axis of the ζ-plane the conditions

$$\operatorname{Re}\frac{\partial\theta}{\partial\zeta} = 0, \qquad \operatorname{Im}\left(m\frac{\partial z}{\partial t}\frac{\partial\bar{z}}{\partial\zeta}\right) - \kappa\operatorname{Re}\frac{\partial z}{\partial\zeta} = \operatorname{Re}\frac{\partial\theta}{\partial\zeta}$$

Let there be a single drain pipe with the center of the cross section at the point $z_0 = ih_0$, the free-surface line coinciding initially with the x-axis, and the flow region being the lower semi-plane. Let $\zeta = hi$ in the ζ -region correspond to the point z_0; here h will change with the time, as will the well output. Put

$$h = h_0 + h_1t + h_2t^2 + \cdots, \; q = q_0 + q_1t + q_2t^2 + \cdots$$

The derivative $\partial\theta/\partial\zeta$ as a function of ζ has the form

$$\frac{\partial\theta}{\partial\zeta} = \kappa i + \frac{q}{2\pi}\left(\frac{1}{\zeta + ih} - \frac{1}{\zeta - ih}\right)$$

The function z is desired in the form of a series

$$z = \zeta + z_1t + z_2t^2 + \cdots \tag{13.16}$$

The following expressions are obtained for z_k and q_k.

$$z_1 = -\frac{q_0}{m\pi}\frac{1}{\zeta + ih_0}, \qquad z_2 = -\frac{A_2}{\zeta + ih_0} - \frac{iB_2}{(\zeta + ih_0)^2},$$

$$z_3 = -\frac{A_3}{\zeta + ih_0} - \frac{iB_3}{(\zeta + ih_0)^2} + \frac{2C_3}{(\zeta + ih_0)^3}$$

$$q_0 = \frac{2\pi\kappa(h_0 + \Delta p/\rho g)}{\log(2h_0/\delta)}, \qquad q_1 = \frac{q_0(3q_0 - 4\kappa\pi h_0)}{4m\pi h_0^2\log(2h_0/\delta)}$$

$$q_2\log\frac{2h_0}{\delta} = \frac{q_1}{8m\pi h_0}\left(\frac{9q_0}{h_0} - 4\kappa\pi\right) + \frac{q_0^2}{8(2m\pi)^2h_0^2}\left(\frac{27q_0}{h_0} - 28\kappa\pi\right) + \frac{(\kappa\pi)^2q_0}{(2m\pi)^2h_0^2}$$

$$A_2 = \frac{1}{(2m\pi)^2}\left(\frac{q_0^2}{h_0^2} + 2m\pi q_1\right), \qquad B_2 = \frac{q_0}{(2m\pi)^2}\left(\frac{q_0}{h_0} - 2\kappa\pi - 2m\pi h_1\right)$$

$$A_3 = \frac{2q_0}{(2m\pi)^3h_0^4}[q_0(q_0 - \kappa\pi h_0) + m\pi(q_1h_0 - q_0h_1)h_0] + \frac{q_2}{3\pi m}$$

$$B_3 = \frac{2q_0}{(2m\pi)^3h_0{}^3}\,[q_0(q_0 - \kappa\pi h_0) + m\pi h_0(q_1h_0 - q_0h_1)] - \frac{\kappa(q_0{}^2 + 2m\pi q_1h_0{}^2)}{3h_0{}^2(2m\pi)^2m} - \frac{q_1h_1 + q_0h_2}{3\pi m}$$

$$C_3 = \frac{q_0}{3h_0{}^2(2m\pi)^3}\,(q_0 - 2\pi\kappa h_0)(q_0 - 2\kappa\pi h_0 - 2m\pi h_1h_0) + \frac{q_0h_1{}^2}{3\pi m}$$

$$h_1 = -\frac{q_0}{2m\pi h_0}, \qquad h_2 = -\frac{q_1}{4m\pi h_0} - \frac{q_0}{2(2m\pi)^2h_0{}^2}\left(\frac{3q_0}{h_0} - \kappa\pi\right) \qquad (13.17)$$

The ordinate of the lowest point of the free surface depends on the time in the following manner:

$$Y_A = \frac{q_0}{m\pi h_0}\,t + \frac{q_0{}^2 + m\pi(q_1h_0 - q_0h_1)h - 2\kappa\pi q_0h_0}{2m^2\pi^2h_0{}^3}\,\tau^2 + \left(\frac{A_3}{h_0} + \frac{B_3}{h_0{}^2} + \frac{2C_3}{h_0{}^3}\right)t^3 + \cdots$$

The problem of a system of drain pipes on an impermeable basis corresponds to the problem of shaft extraction of oil procedures. This problem was studied by Kalinin.

4. Problem of the Displacement of a Liquid Surface

In a steady flow we assume in the flow region at the time $t = 0$ an arbitrary surface consisting of definite liquid particles. How does the surface move? In this way one may set up the problem of the displacement of a water-oil interface if one assumes that the viscosities and permeabilities of water and oil are equal. Plane problems in such a formulation have been studied by Muskat and Shchelkachev. A three-dimensional case is investigated by Millionshchikov (87).

Let the equation of the moving liquid surface be $F(x,y,z,t) = 0$. Differentiating with respect to t, we get

$$\frac{\partial F}{\partial x}\,v_x + \frac{\partial F}{\partial y}\,v_y + \frac{\partial F}{\partial z}\,dz + m\,\frac{\partial F}{\partial t} = 0 \quad \left(\frac{dx}{dt} = \frac{1}{m}\,v_x, \;\cdots\;\right)$$

We substitute for v_x, v_y, v_z their expressions for the case of the given flow, integrate the partial differential equation in F, and find its particular solution for the given initial condition. Let for $t = 0$ be $z = -h$.

As a flow model Millionshchikov takes a well in an infinite space, for

$$\varphi = -\frac{Q}{2\pi r} \quad (r = \sqrt{x^2 + y^2 + z^2})$$

Passing to cylindrical coordinates ρ, z, we get

$$v_\rho = \frac{\partial \varphi}{\partial \rho} = \frac{Q}{2\pi}\frac{\rho}{r^3}, \qquad v_z = \frac{\partial \varphi}{\partial z} = \frac{Q}{2\pi}\frac{z}{r^3}$$

The differential equation for F becomes

$$\frac{Q}{2\pi r^3}\left(\rho \frac{\partial F}{\partial \rho} + z \frac{\partial F}{\partial z}\right) + m \frac{\partial F}{\partial t} = 0 \quad (r^2 = \rho^2 + z^2)$$

The solution we are seeking is

$$z^3(\rho^2 + z^2)^{3/2} + \frac{3Q}{2\pi m} t = h^3(z^2 + \rho^2)^{3/2} \tag{13.18}$$

The moment of the inrush of the bottom water into the well is

$$t = \frac{2\pi h^3 m}{3Q} \tag{13.19}$$

After the inrush, the water will enter into the well with the oil, and the water cone near the well outlet will be surrounded by an annular region of oil. Designating by α the angle which the tangent to the interface encloses with the horizontal plane (top of the layer), we get

$$\sin \alpha = \frac{2\pi h m}{3Qt} \tag{13.20}$$

Now the author gets an expression for the "coefficient of oil," i.e., the ratio of the oil discharge to the total output of the well.

5. The Motion of Two Liquids with Different Densities and Viscosities

This may be studied approximately by means of the Fourier integral. As an example let us take the flow of oil of density ρ_1 and viscosity μ_1, filling, at the initial moment, a strip $0 < y < h_1$, and water of density ρ_2 and viscosity μ_2 in the strip $-h_2 < y < 0$. At the point $z = h_1 i$, there is a sink of intensity Q. Then for the initial period of filtration, as long as the interface differs little from a horizontal plane, one gets the equation of this surface in the form

$$y = \frac{Q}{2\pi\epsilon_1}\int_0^\infty \frac{\cos \alpha x}{\alpha \sinh \alpha}(e^{\sigma t - 1})d\alpha$$

where

$$\sigma = \frac{\alpha \sinh \alpha h_1 \sinh \alpha h_2}{\left(\dfrac{m \sinh \alpha h_1 \cosh \alpha h_2}{\epsilon_1} + \dfrac{\cosh \alpha h_1 \sin \alpha h_2}{\epsilon_2}\right)}$$

$$\epsilon_1 = \frac{\kappa g}{\mu_1}(\rho_2 - \rho_1), \qquad \epsilon_2 = \frac{\kappa g}{\mu_2}(\rho_2 - \rho_1)$$

6. *Elastic Regime*

If the water, under whose action the inflow of the oil to the wells takes place occupies a large volume, the compressibility of the liquid plays an important role, as has been proved in the case of the East Texas oil fields (88). One had to assume the value of the compressibility coefficient to be very large to be able theoretically to explain the observed phenomena. Shchelkachev (157,159,160) considers, besides the elasticity of the water, also the elasticity of the layer consisting of soil grains. The pressure in the elastic regime satisfies approximately the heat transfer equation

$$a^2\Delta p = \frac{\partial p}{\partial t}$$

According to Shchelkachev the coefficient a^2 depends on the compressibility coefficient β_e of the liquid and on the soil skeleton β_s of the layer. Hence

$$a^2 = \frac{k}{\mu(m\beta_e + \beta_s)} \tag{13.21}$$

where m is the porosity, and μ the dynamic viscosity of the liquid.

The author (126) applied the general theory to the actually observed case of interference of two wells.

References

The original reference style has been retained as presumably the most effective for the purpose of identification to Russian libraries. The abbreviations which the translator has been able to decode are explained below, as well as the adopted transliteration system. The latter determines uniquely the Russian spelling.

DAN Doklady Akademii Nauk.
Izv OTN AN Izvestiya, Otdel Tekhnicheskikh Nauk Akademiya Nauk.
PMM Prikladnaya Matematika i Mekhanika.
ZAMM Zeitschrift für angewandte Mathematik und Mechanik.

1. Aravin, V. I., On filtration from reservoirs, *Izv. NIIG*, XVI, 104–111 (1935).
2. ——, Inflow of soil waters to water collectors, *Izv. NIIG*, XVIII, 4–43 (1936).
3. ——, On the question of seepage in anisotropic-impermeable soils, *Trudi Leningr. industr. in-ta. Razdel gidrotekhniki*, No. 19, 2–12 (1937).
4. ——, Inflow of soil waters to a ditch enclosed by cutoffs, *Izv. NIIG*, XX, 74–89 (1937).
5. ——, Inflow of soil waters to a ditch of rectangular cross section, resting on a horizontal impermeable stratum. *Izv. NIIG*, XXII, 81–92 (1938).
6. ——, Calculation of seepage around hydrotechnical installations, *Izv. NIIG*, XXVII (1940).
7. Bagrov, N. A., Some problems in unstable flow of soil waters, Dissertation, Moskov. gidrometeorolog. Institut., 1938.

8. Bazanov, M. I., Study of seepage in the case of water inflow to dessicating wells, *PMM*, **II**, No. 2, 223–241 (1938).
9. ——, On seepage under jumps in conditions of a plane problem, *Izv. NIIG*, **XXVIII**, 116–131 (1940).
10. Bildyug, E. I., Calculation of seepage of an earth dam with a shield and cutoff, *Izv. NIIG*, **XXVIII**, 227–284 (1940).
11. Braginskaya, G. A., Flow around an apron in an anisotropic soil, with seepage, of two liquids of different densities, *PMM*, **VI** (1942).
12. Charny, I. A., On a method of evaluation of industrial reserves of oil fields, *Izv. OTN AN SSSR*, No. 11–12, 1943.
13. ——, On oil inflow to wells in oil fields of circular and strip form, *DAN*, **XLII**, No. 4 (1944).
14. ——, On oil inflow to wells in oil fields of oval or sickle form, *DAN*, **XLII**, No. 5 (1944).
15. ——, Influence of compressibility of marginal waters on the working regime of oil fields, *Izv. OTN AN SSSR*, No. 7–8, 1944.
16. ——, On the most favorable distribution of wells in oil layers under water pressure conditions, *Izv. OTN AN SSSR*, No. 1–2, 1945.
17. ——, On the interference of imperfect wells, *Izv. OTN SSSR*, No. 11, 1946.
18. Darcy, H., Les fontaines publiques de la ville de Dijon, 1856.
19. Davison, B. B., On plane pressureless steady flow of soil waters, Trudĭ IV gidrol. konferentsii Baltiĭskikh stran. Leningrad, vol. IX, 1933.
20. ——, Plane steady flow of soil waters with free surface, Trudĭ IV gidrol. konferentsii Baltiĭskikh stran. Leningrad, vol. 14, 1933.
21. ——, (Jointly with Khristianovich, S. A.) On some problems of theoretical hydrology, *Uspekhi matem. nauk*, No. 2, 245, 1936.
22. ——, Some cases of plane steady motion of soil waters in the presence of evaporation, *Trudĭ Gos. gidrol. inst.*, No. 5, 183–192, 1937.
23. ——, On the question of a simplified method of computation of forced flow of soil waters, *Trudĭ Gos. gidrol. in-ta*, No. 5, 194–199, 1937.
24. ——, Some exact solutions of the problem of the flow of soil waters obtainable by a simplified method, *Izv. OTV AN*, No. 8–9, 59–64, 1938.
25. ——, On unsteady flow of soil waters without free surface, *Trudĭ Gos. gidrol. inst.*, No. 5, 212–217, 1937.
26. ——, On the steady two-dimensional motion of ground water with a free surface. *Phil. Mag.*, **21**, 881 (1936).
27. ——, On the steady motion of ground water through a wide prismatic dam, *Phil. Mag.*, **21**, 104 (1936).
28. ——, (And Rosenhead, L.) Some cases of the steady two-dimensional percolation of water through ground, *Proc. Roy. Soc. (London)*, No. **962**, 366–381 (1940).
29. ——, Motion of ground waters. Part 3 of the book: Khristianovich, S. A. Mikhlin, S. G., Davison, B. B. Some new questions of mechanics of a compact medium. Izd. Akad. Nauk SSSR.
30. Dupuit, J., Études théoriques et pratiques sur le mouvement des eaux dans les canaux découverts et à travers les terrains perméables, 2nd ed., Paris, 1863.
31. Falkovich, S. B., Some problems of steady flow of soil waters. Dissertation Saratov. Saratovskiĭ gosudarstvenniĭ universitet, 1941.
32. Fandeev, V. V., Determination of forces in the calculation of the stability of slopes of earth dams. Gidrotekhn. stroitelstvo, No. 1, 1947.
33. Forchheimer, Ph., Hydraulics.

34. ——, Zur Theorie der Grundwasserströmung. Sitzungsberichte d. Wiener Akademie, 128 [8], 1245 (1919).
35. ——, Über die freien Spiegel bei Grundwasserströmungen, *ZAMM*, 2, [1] (1922).
36. Galin, L. A., Unsteady seepage with free surface, *DAN*, XLVII, No. 4, 250–253 (1945).
37. Gersevanov, N. M., Applications of functional analysis to the solution of problems on filtration of soil waters, *Izv. OTN AN*, No. 7, 73–89 (1943).
38. ——, Principles of dynamics of the soil mass. M.-L., 1937.
39. Girinsky, N. K., Principles of the theory of motion of soil waters under hydrotechnical installations in the presence of a constant pressure along the lower soil surface, *Gidrotekhn. stroitelstvo*, No. 6, 22–28, 1936.
40. ——, Soil flow of earth dams, *Nauchnie zap. Mosk. inst. inzh. vodnogo khoz.* 4, 1–38 (1937).
41. ——, Principles of the theory of motion of soil waters under hydrotechnical installations in heterogeneous soils, *Nauchn. zap. Mosk. gidro-melior. inst.*, No. 5, 47–100, 1938.
42. ——, Graphical construction of hydrodynamic grids in the case of seepage in homogeneous soils. *Nauchn. zap. Mosk. gidro-melior. inst.*, VII, No. 11, 3–22 (1939).
43. ——, Coefficient of seepage of capillary water, *Gidrotekhn. stroitelstvo*. No. 1, 1939.
44. ——, Seepage discharge under hydrotechnical installations on heterogeneous soils, Stroĭizdat, 1941.
45. ——, Complex potential of a flow with free surface in a stratum of relatively small output at $k = f(v)$. *DAN*, LI, No. 5, 337–338 (1946).
46. ——, Generalization of some solutions for wells to more complicated natural conditions, *DAN*, LIV, No. 3 (1946).
47. ——, Some questions of dynamics of underground waters, Sbornik Vseginгeo, No. 9, 1947.
48. Grib, A. A., Steady flow of soil waters in the presence of a drain pipe, free surface and an impermeable wedge-shaped stratum, *Izv. NII mat. i mekh. pri Tomskom gos. universitete*, II, No. 2, 192–219 (1938).
49. Hamel, G., Über Grundwasserströmungen, *ZAMM*, 1934, XIV [3], 129–159 (1934).
50. ——, (And Günther, E.) Numerische Durchrechnung zu der Abhandlung über Grundwasserströmungen, *ZAMM*, XV, 255–265 (1921).
51. Hopf, L., and Trefftz, E., Grundwasserströmung in einem abfallenden Gelände mit Abfanggraben, *ZAMM*, I, [4], 290–297 (1921).
52. Ivakin, V. V., On seepage accompanying pumping into a soil unsaturated with water, *Izv. OTN AN SSSR*, No. 6, 669–680, 1947.
53. Izbash, S. V., On seepage in coarse-grained material, *Izv. NIIG*, 1, 1–48 (1931).
54. Joffe, A. Ya., On the question of water seepage in a heterogeneous medium, *Zhurnal tekhnicheskoi fisiki*, V, No. 1 (1935).
55. Kalinin, N. K., Some approximate methods for the problem of seepage in a two-layer medium, *DAN*, XXX, No. 7, 593–597 (1941).
56. ——, On the question of seepage in a multilayer soil, Dissertation, Saratov states university, 1943.
57. ——, Reduction of some problems on seepage in a two-layer medium to seepage problems in a homogeneous soil, *DAN*, X, No. 5, 204–207 (1943).
58. ——, On the method of P. Ya. Kochina for problems on the motion of soil waters, *DAN*, XLV, 110 (1944).

59. ——, (Jointly with POLUBARINOVA-KOCHINA, P. YA.) On unsteady motion of soil waters with a free surface, *PMM*, XI, No. 2, 231–236 (1947).
60. KAMENSKY, G. N., KORCHEBOKOV, N. A., RAZIN, K. I., Motion of ground waters in heterogeneous strata, *ONTI*, 1935.
61. KAZARNOVSKAYA, B. E., (And POLUBARINOVA-KOCHINA, P. YA.) On the motion of foot waters in oil strata, *PMM*, VII, No. 6, 439–454 (1943).
62. ——, Displacement of the water-oil interface and the infiltration of wells caused by water pressure on oil deposits, *DAN*, LV, No. 8, 603–696 (1947).
63. KHOMOVSKAYA, E. D., Hydromechanic solution of the problem of inflow of soil waters to a water-collecting pipe in the presence of a very heavy impermeable stratum. *Trudi Gos. Gidrol. inst.*, No. 5., pp. 200–211, 1937.
64. KHRISTIANOVICH, S. A., (Jointly with B. B. DAVISON). See Davison (21).
65. ——, Soil water motion violating the Darcy law. *PMM*, IV, No. 1, 33–52 (1940).
66. ——, On the motion of a gas-liquid mixture in porous media, *PMM*, V, No. 2, 277–282 (1941).
67. KOZENY, J., Grundwasserbewegung bei freiem Spiegel, Fluss und Kanalversicherung, *Wasserkraft und Wasserwirtschaft*, No. 3, 1931; No. 8–9, 1934.
68. KOZLOV, V. S., Determination of hydrodynamic elements for a triple-cutoff apron on a permeable foundation of finite depth, *DAN*, XVI, No. 1 (1937).
69. ——, Hydromechanical computation of cutoff dams, *Izv. OTN AN SSSR*, No. 6, 89–110, 1939.
70. ——, On the computation of water motion under hydrotechnic installations in anisotropic permeable soils, *Izv. OTN AN*, No. 3, 59–79, 1940.
71. ——, Determination of the hydrodynamic elements in a triple-cutoff apron, the inner cutoff being nonsymmetrically situated, *Izv. OTN AN*, No. 6, 58–72, 1940.
72. ——, Computation of drainage installations, Stroĭizdat, 1940.
73. ——, Hydromechanical computation of aprons, Gosènergoizdat, 1941.
74. KUFAREV, P. P., VINOGRADOV, YU. P., On some particular solutions of the seepage problem, *DAN*, LVII, No. 4 (1947).
75. KUZNETSOV, V. N., On the motion of soil waters, *Trudi MIIT*, No. 6, 1927.
76. LAVRENTIEV, M., Conformal mapping, *OGIZ*, 1946.
77. LAVRENTIEV, M., POGREBISSKY, U. B. On the question of the motion of soil waters in a hetereogeneous soil, *Dopovidi AN URSR*, No. 1, 1940.
78. LEIBENSON, L. S., Mechanics of the oil industry, Part 2, 1934.
79. ——, Motion of a mixture of gas and liquid in a porous medium, *Izv. AN. Seriya geofiz.*, No. 4–5, 411–422, 1941.
80. ——, Principles of the laws of motion of a gas in a porous medium, *DAN*, 1945, XLVII, No. 1, 1945.
81. ——, New equations of motion of a mixture of gas and liquid in a porous medium, *DAN*, XLIX, No. 3, 173–176 (1945).
82. ——, Motion of natural liquids and gases in porous media, *GONTI*, 1947.
83. MELESHCHENKO, N. T., Computation of soil water motion under installations having drain apertures, *Izv. NIIG*, No. 19, 25–47, 1936.
84. ——, Computation of soil water flow in the case of a flat apron with an inclined impermeable base stratum, *Izv. NIIG*, No. 18, 44–49, 1936.
85. ——, Flow of soil waters under hydrotechnic structures (methods of computation), 1937.
86. ——, An approximate method of seepage study under structures situated on an infinite permeable foundation, *Izv. NIIG*, No. 28, 234–272, 1940.

87. Millionshchikov, M. D., Hydromechanic analysis of some methods of operation of oil wells, Dissertation, Institut mekhaniki Akademii Nauk SSSR. Moskva, 1944.
88. Muskat, M., The flow of homogeneous fluids through porous media. New York-London, 1937.
89. Myatiev, A. N., Operation of a well in a supply basin of soil waters, *Izv. Turkmenskogo filiala AN SSR*, No. 3-4, 43-50, 1946. Also *Izv. OTN*, No. 9, 1947.
90. Nedriga, V. P., Computation of seepage around hydrotechnic structures, Gidrotekhn. stroitelstvo, 1947, No. 5.
91. Nelson-Skornyakov, F. B., Computation of the flow of soil waters through earth dams, Izd. Vsesoyzn. akad. s.-kh. nauk im. Lenina, 1936.
92. ——, Flow of soil water through earth dams on permeable foundations, *Nauchnie zap. Mosk. inst. inzh. vodnogo khoz*, No. 4, 77-89, 1937.
93. ——, On the flow of soil water through drained earth dams, *Gidrotekhn. stroitelstvo*, No. 2, 32-36, 1937.
94. ——, Flow of soil water with a free surface through earth dams with a diaphragm, core and screen on permeable foundations, *Gidrotekhn. stroitelstvo*, No. 7-8, 25-30, 1937.
95. ——, Flow of soil water through a dambridge, drained at the foundation, *Izv. OTN AN SSSR*, No. 6, 81-84, 1940.
96. ——, Flow of soil water through an earth dam with central diaphragm, *Izv. OTN AN SSSR*, No. 7, 53-64, 1940.
97. ——, Inflow of soil water to drain channels on a rock boundary, *DAN*, XXVIII, No. 6, 483-487 (1940).
98. ——, An earth dam on a permeable foundation of finite depth with a nonhorizontal impermeable base stratum, *DAN*, XVIII, No. 6, 488-493 (1940).
99. ——, Flow of soil water with a free surface through an earth dam with a central diaphragm and nonhorizontal rock boundary, *Izv. OTN AN SSSR*, No. 1, 119-125, 1941.
100. ——, Inflow of soil water with free surface to a system of drains with deep rock boundary, *Izv. OTN AN*, No. 1, 126-128, 1941.
101. ——, Flow of soil water with free surface through a dam with two diaphragms, *Izv. OTN AN SSR*, No. 3, 39-46, 1941.
102. ——, A hydromechanical method for the infiltration of soil waters to a ditch, *Izv. OTN AN*, No. 7, 90-94, 1943.
103. ——, Seepage in homogeneous medium, Moskva, 1947.
104. Numerov, S. N., On seepage in earth dams with drainage on impermeable foundations. *Izv. NIIG*, XXV (1939).
105. ——, On the effect of infiltration or evaporation of water from the free surface in the computation of soil water flow, *PMM*, IV, No. 5-6, 11-18 (1940).
106. ——, On seepage through earth dams with drainage on permeable foundations, *Izv. NIIG*, XXVII (1940).
107. ——, On the question of seepage through earth dams, *Izv. NIIG*, XXVIII, 64-90 (1940).
108. ——, A hydromechanical method of computation of a bank drain, *Izv. NIIG*, XXVIII, 91-108 (1940).
109. ——, A practical method of computation of seepage for earth dams with drainage on impermeable foundations, *Izv. NIIG*, XXVIII, 272-276 (1940).
110. ——, Solution of problems in seepage without seepage segments and without infiltration or evaporation of water from free surfaces, *PMM*, VI (1942).

111. ——, A practical method of seepage computation for earth dams with drainage on an impermeable foundation in the presence of tail water. *Izv. NIIG*, XXXI, pp. 276 (1946).
112. Pavlov, A. T., Flow around a cutoff in a two-layer fluid. *PMM*, VI, 1942.
113. Pavlovsky, N. N., Theory of soil-water flow under hydrotechnical structures and its basic applications, Petrograd, 1922.
114. ——, On water seepage through earth dams, 1931.
115. ——, Principles of hydromechanical computation of dams. Senkov type. *Gidrotekhn. stroitelstvo*, No. 8–9, 1936.
116. ——, Principles of the method of hydromechanical solution of problems of free filtration from open channels, *Izv. inst. gidrotekhniki*, XIX, 5–24 (1936).
117. ——, Free filtration from open channels with a circular base, *Izv. NIIG*, XIX, 49–81 (1936).
118. ——, On the application of the Kirchhoff complex to the solution of problems of pressureless filtration, *Izv. NIIG*, XX, 5–25 (1937).
119. ——, On water inflow to horizontal filters, *Izv. NIIG*, No. 21, 5–22, 1937.
120. Pykhachev, G. V., On the output of wells in a layer of nonhomogeneous permeability. *Trudi Grozn. neftyan. inst.*, vip. 1, 1944.
121. ——, On the question of displacement of oil by water in a layer of nonhomogeneous permeability, *Trudi Grosn. neftyan. inst.*, No. 2, 1944.
122. Pogrebissky, U. B., On the question of water seepage through earth dams, *Visti Institutu vodnogo gospodarstva*, 2, 181–191 (1935).
123. Polubarinova-Kochina, P. Ya., Application of linear differential equations to some cases of soil-water flow, *Izv. AN. Otd. mat. i est. navk, seriya mat.*, No. 3, 371–395, 1938.
124. ——, Application of the theory of linear differential equations to some problems in the flow of soil waters. (The case of three singularities), *Izv. AN, seriya mat.*, No. 3, 329–350, 1939.
125. ——, On the continuous change of the velocity hodograph in a plane steady flow of soil waters, *DAN*, XXIV, No. 4, 1327–1330 (1939).
126. ——, The simplest cases of soil-water flow in two layers with different seepage coefficients, *Izv. OTN AN SSSR*, No. 6, 75–88, 1939.
127. ——, Example of soil water flow through an earth dam in the presence of evaporation, *Izv. OTN AN SSSR*, No. 7, 45–52, 1939.
128. ——, Application of the theory of linear differential equations to some problems on soil water motion (the number of singularities exceeding three), *Izv. AN SSSR, seriya mat.* No. 5–6, 579–602, 1939.
129. ——, On unsteady motion of soil waters in two strata of different densities, *Izv. OTN AN SSSR*, No. 6, 73–80, 1940.
130. ——, Computation of the filtration through an earth dam, *PMM*, IV, No. 1, 53–64 (1940).
131. ——, On filtration in an isotropic soil, *PMM*, IV, No. 2, 101–104 (1940).
132. ——, On the question of soil water motion in a drained dam, *Uchenie zap. Mosk. gos. un-ta*. No. 39, mekhanika, 91–102, 1940.
133. ——, On the question of filtration in a two-layer medium. *DAN*, XXVI, No. 4, 339–341 (1940). (The error contained in this paper is corrected by N. K. Kalinin in *Fiz.-mat. referat. zhurnal*, IV, No. 4, 1622 (1940).)
134. ——, On the simplest cases of filtration in a two-layer medium. *DAN*, XXXVI, No. 8, 746–749 (1940).
135. ——, On filtration in a multi-layer medium, *Inzh. sbornik.*, No. 2, 1940.

136. ——, On filtration under hydrotechnic structures in a multilayer medium, *PMM*, No. 2, 1941.
137. ——, Some problems of plane soil water motion, *Izd. Akademii Nauk SSSR*, 1–142, 1942.
138. ——, On liquid inflow to wells in a heterogeneous medium, *DAN*, XXIV, No. 2, 46–51 (1942).
139. On the direct and inverse problems of the hydraulics of an oil layer, *PMM*, VII, No. 5, 361–374 (1943).
140. ——, See KAZARNOVSKAYA, B. E.
141. ——, On the displacement of the oilfield boundary, *DAN*, XLVII, No. 4, 254–257 (1945).
142. ——, On unsteady motions in the theory of filtration, *PMM*, IX, No. 1, 76–90 (1945).
143. ——, (Jointly with N. K. KALININ) see KALININ, N. K.
144. ——, On the hydraulic theory of wells in a multilayer medium, *PMM*, XI, No. 3, 357–362 (1947).
145. RISENKAMPF, B. K., Hydraulics of soil waters. Part 1. *Uchen. zap. Saratovskogo gos. un-ta. Seriya, fiz.-mat.*, 1 (XIV), No. 1, 89–114 (1938).
146. ——, Hydraulics of soil waters. Part 2. *Uchen. zap. Saratovskogo gos. un-ta. Seriya mat. mekh.* 1 (XIV), No. 2, 181–205 (1938).
147. ——, Hydraulics of soil waters. Part 3. *Uchen. zap. Saratovskogo gos. un-ta.* XV, No. 5, Gidravlika, 3–93 (1940).
148. ——, On a case of filtration of water in a multilayer soil, *Uchen. zap. Saratovskogo gos. un-ta*, XV, Gidravlika, 94–100 (1940).
149. RISENKAMPF, B. K., KALININ, N. K., On the motion of soil waters in three dimensions with an elipsoidal free surface, *PMM*, V, No. 2, 283–286 (1941).
150. ROSSBACH, H., Uber Grundwasserströmung, *Ing. Arch.*, No. 1,3,5, 1936, No. 2, 1938.
151. SEGAL, B. I., (Jointly with F. R. GANTMAKHER) A method of hydromechanic computation of a system of dams, *DAN*, XXXV, No. 4, 103–109 (1942).
152. ——, Approximate calculation of some hyperelliptic integrals encountered in the computation of dams, *DAN*, No. 7, 223–229 (1942).
153. ——, Computation of seepage under dams with permeable foundation of infinite depth, *Inzh. sbornik*, III, No. 1.
154. ——, Some spatial problems of the potential theory and its applications, *Izv. AN SSSR, Seriya matem*, 10, 323–358, 1946.
155. SHEKHTMAN, YU. M., Numerical and experimental verifications of seepage computations by the method of conformal mapping and the method of fragments, *Inzh. sbornik*, 1, No. 1 (1941).
156. SHCHELKACHEV, V. N., (Jointly with V. G. PYKHACHEV) Interference of wells and the theory of stratified water-pressure systems, Baku, 1939.
157. ——, Principles of underground oil hydraulics, Gostoptekhizdat, 1945.
158. ——, Basic equations of the motion of an elastic liquid in an elastic porous medium, *DAN*, II, No. 2 (1946).
159. ——, Study of a one-dimensional motion of an elastic liquid in an elastic porous medium, *DAN*, II, No. 3 (1946).
160. ——, Some applications of the theory of the elastic regime in water-oil bearing layers, *DAN*, II, No. 5 (1946).
161. SLICHTER, CH., Theoretical investigations on the motion of ground waters, *19th Ann. Rep. U.S. Geol. Survey*, p. II, 1897–1898.

162. Topolyansky, D. B., On the problem of the motion of soil waters in a heterogeneous ground in the presence of drain sinks (in Ukrainian). *Dopovidi. AN URSR*, No. 7, 1940.
163. ——, On the motion of soil waters in a heterogeneous ground in the presence of drain sinks, *Zbirnik prats Institutu matematiki AN URSR*, No. 8, 1947.
164. Uginchus, A. A., New formulas for the computation of seepage through earth dams, *Gidrotekhn. stroitelstvo*, No. 5, 1–6, 1947.
165. Vedernikov, V. V., Seepage from channels, Gosstroĭizdat, 1934.
166. ——, Influence of capillary rise on seepage from channels, *Gidrotekhn. stroitelstvo*, No. 9, 20–27, 1935.
167. ——, Method of solution of some problems on seepage with free surface, *Gidrotekhn. stroitelstvo*, No. 9, 12–14, 1935.
168. ——, Seepage from triangular and trapezoidal channels, *Nauchnie zap. Mosk. instituta inzh. vodnogo khoz.*, No. 2, 248–288, 1936.
169. ——, Influence of capillarity on seepage with a free surface, *DAN*, III, No. 4 (1936).
170. ——, Hydromechanical methods of computation of the flow of soil waters with a free surface, *Nauchn. zap. Mosk. insta. inzh. vodnogo khoz.*, No. 4, 1937.
171. ——, On the theory of drainage, *DAN*, XXIII, No. 4, 335–337 (1939).
172. ——, Theory of seepage and its applications to problems of irrigation and drainage, Gosstroĭizdat, 1939.
173. ——, An estimate of the influence of soil capillarity on the seepage from channels, *DAN SSSR*, XXVIII, No. 5, 408–410 (1940).
174. ——, Seepage through earth dams on permeable foundation, *DAN*, L, 107–110 (1945).
175. ——, Computation of seepage through earth dams, *Gidrotekhn. stroitelstvo*, No. 1, 12–15, 1947.
176. ——, Influence of capillary rise on seepage from channels, *Gidrotekhn. stroitelstvo*, No. 5, 1935.
177. —— On the question of seepage through earth dams, *Irrigatsiya i gidroteknika*, No. 5, 1936.
178. ——, On the theory of drainage. Contributions to the increase of the productivity of agricultural lands by means of melioration, *Severniĭ nauchno-issl. inst. gidrotekhniki i melioratsii*, No. 4, 1935.
179. ——, Improvement of foundations by silt deposition accompanying the filtration of a water mass, *DAN*, XXV, No. 1 (1942).
180. ——, Seepage through an earth dam on permeable foundation, *DAN*, L (1945).
181. ——, Results of studies of the physical picture of free seepage, *DAN*, LX, No. 3 (1947).
182. ——, Results of experiments on free seepage, *Izvestiya OTN AN*, No. 8, 1947.
183. ——, Der Einfluss des kapillaren Aufstieges auf die Sickerung aus Kanalen, *Wasserkraft und Wasserwirtschaft*, No. 21, 1935.
184. ——, Über die Sickerung und Grundwasserbewegung mit freier Oberfläche, *ZAMM*, XVII [3], (1936).
185. Verigin, N. N., Seepage through the foundation of dams with oblique screens and cutoffs, *Gidrotekhn. stroitelstvo*, No. 2, 30–33, 1940.
186. ——, Seepage around dams and the efficiency of antiseepage screens, *Gidrotekhn. stroitelstvo*, No. 5, 10–14, 1947.
187. Vinogradov, Yu. P., and Kufarev, P. P. (see Kufarev).
188. Voshchinin, A., Motion of soil waters in the body and foundation of a homo-

geneous earth dam with a horizontal filter at finite depth of the permeable foundation, *DAN*, XXV, No. 9, 733–735 (1939).

189. Zamarin, E. A., Determination of motion of ground waters, Tashkent, 1928.
190. Zamarin, E. A., and Shipenko, N. I., Nomographic computations of hydro-installations, M., 1934.
191. Zamarin, E. A., Hydrodynamic motion grids. *Nauchnie zap. MIIVKh.* No. 4, 1937.
192. ——, Seepage through earth dams, *Nauchnie zap MGMI*, VII, No. 1 (1939).
193. ——, Water losses through seepage from reservoirs, *Nauchnie zap. MGMI*, VII, No. 8 (1939).
194. Zauerbrei, I. I., On the question of the seepage coefficient of soils and the methods of its investigation, *Izv. NIIG*, II, ch. 1, pp. 115–145; V, ch. 2, pp. 173–206, 1932.
195. Zhukovsky, N. E., Theoretical investigation on the motion of ground waters, 1889, Poln. sobr. soch. 1937, VII, 9–33.
196. ——, On the influence of the pressure on water-saturated sands (1890), Poln. sobr. soc. VII, 34–42.
197. ——, Seepage of water through dams (1932), Poln. sobr. soch., VII, 323–363.

Author Index

Numbers in parentheses are reference numbers. They are included to assist in locating references in which the authors' names are not mentioned in the text. *For example:* Frank, Ph., 28(3), *91*, indicates that reference 3 on page 28 is to this author's article. Numbers in italics refer to the page at the end of the chapter on which the reference is listed.

Numbers in bold-face type indicate the pages of the chapter in this volume written by the author mentioned. *For example:* Clark, R. A., **93–122** means that this author wrote the chapter included in pages 93–122.

A

Airy, J. L., 136, 137
Aravin, V. I., 158, 165, *217*

B

Bäcklund, B., 132
Bagrov, N. A., *217*
Ballabh, R., 129, *149*
Bantlin, A., 93, *122*
Batchelor, G. K., 1, 5, 6, 8, 13, 15, *19*
Bateman, H., 126, 131, *149*, 197
Bazanov, M. I., 162, *218*
Bechert, K., 135, *150*
Beltrami, E., 125, 126, 129, 134, *148*
Berker, R., 126, 127, 128, 129, 133, *148*
Beskin, L., 95, 102, 104, 105, 108, 113, 114, 115(2), 116(2), *122*
Betti, E., 145
Bildyug, E. I., *218*
Birkhoff, Garrett, 148
Blasius, H., 29, 30, 31, 38, 41, 45, 48, 58, 60, 63, 64, 67, 68, 72, 78, 81, 85, 88, *91*
Boussinesq, J., 129, 136, 137, 142
Braginskaya, G. A., *218*
Brainerd, G. J., 32, 37(14, 15), 60, 63, 64(14, 15), 65(15), 67, 69(14), 78, 79, 83(14, 15), *91*, *92*
Busemann, A., 28(8), 30(8), 70(8), 77, *91*

C

Caldonazzo, B., 129
Carathéodory, C., 142, 143, 145, 147
Carothers, S. D., 136, *150*
Carrier, G. F., 28(28), 84, 89(28), *92*
Castoldi, L., 125, *148*
Chaplygin, S. A., 132, 146
Chapman, D. R., 37, 38, 39(21), 40, 41, 42(19), 43(19), 50, 86, 87(19), 89, *92*
Chapman, S., 26
Charny, I. A., 202, 203, 204, 210, *218*
Chiarulli, P., 23
Christensen, S., 91
Clark, R. A., **93–122**
Cope, W. F., 25(18), 28(18), 37(18), 40(18), 46, 54(18), 57(18), 60, 62(18), 67(18), 82(18), 83(18), *92*
Coulomb, C. A., 141
Courant, R., 131, *149*
Cowling, T. G., 26
Crocco, L., 30, 32, 36(16), 37, 42, 43, 44, 49, 54, 70, 72, 73(16), 75, 76, 78, 83(16), 85, *91*, *92*

D

Darcy, H., 154, 155, 166, 172, 193, 204, *218*
Davison, B. B., 155, 156, 157, 172, 176, 210, 211, *218*, *220*
Dorodnitsyn, A. A., 50
Doucet, E., 129, 136, *150*
Dougall, J., 137
Drewitz, O., 64, 66(22), 84, 86, 88(22), *92*
Dryden, H. L., 7, 8, *19*
Dupuit, J., 201, *218*
Dworzak, W., 138, *150*

E

Eckert, E., 64, 66(22), 84, 86, 88(22), *92*
Emmons, H. W., 32, 37(14, 15), 60, 63, 64(14, 15), 65(15), 67, 69(14), 78, 79, 83(14, 15), *91*, *92*

F

Falkner, V. M., 67, 88, 90
Falkovich, S. B., **153–225**, 168, 169, 171, 173, 178, 180, 181, 195, 196, *218*
Fandeev, V. V., 171
Föppl, L., 137, *150*
Forchheimer, Ph., 201, *218*
Frank, Ph., 28(3), *91*
Frankl, F., 54
Freeman, J. C., 23
Frenkiel, F. N., 1, *19*
Friedrichs, K. O., 131, *149*

G

Galin, L. A., 211, *219*
Gantmakher, F. R., *223*
Gauster, W., 141, 147, *151*
Geiringer, H., 141(48), 143, *151*
Gersevanov, N. M., 191, 192, *219*
Giese, J. H., 132, 135(23), *149*, *150*
Gilbarg, D., 125, 147, *148*
Girinsky, N. K., 162, 171, 199, *219*
Görtler, H., 127, *149*
Goldstein, S., 25(1), 28(1), 35(1), 51(1), 53(1), 54(1), 56(1), 57(1), 58(1), 66(1), 81(1), *91*
Gray, A., 197
Grib, A. A., 171, *219*
Günther, E., *219*
Guenther, P. E., 133

H

Hadamard, J., 135, *150*
Hamel, G., 125, 127, 147, *148*, *149*, 176, *219*
Hantzsche, W., 32, 37, 42, 45, 70, 82, 83(13), 84, *91*
Harms, F., 30(6), *91*
Hartree, D. R., 25(18), 28(18), 37(18), 40(18), 46, 54(18), 57(18), 60, 62(18), 67(18), 82(18), 83(18), 88, *92*
Heisenberg, W., 1, 5, 8, 13, *19*
Helmholtz, H. von, 147
Hencky, H., 141
Hicks, B. L., 133
Hopf, L., 183, *219*
Howarth, L., 1, 2, 3, 4, 6, 8, 9, *19*, 32, 37, 40, 46, 47, 48, 50, 51, 55, 56(17), 57, 58(17), 59, 72, 89(17), 90, *92*
Huber, M. T., 95, 118, *122*

I

Illingworth, C. R., 90, 91
Ivakin, V. V., 206, *219*
Izbash, S. V., *219*

J

Jeffery, G. B., 126, 127, 147, *148*
Joffe, A. Ya., *219*
Johnson, H. A., 82(27), 83, *92*

K

Kalikhman, B. L., 37, 48, 50, 51, 54(20), *92*
Kalinin, N. K., 178, 196, 199, 206, 211, 213(59), 215, *219*, *220*, *222*, *223*
Kamensky, G. N., *220*
Kampé de Fériet, J., 3, *19*, 128, 129, 147, *149*
Karl, H., 95, 102, 104, 105, 107, 108, *122*
Kármán, Th. von, **1–19**, 1, 2, 3, 4, 5, 6, 8, 9(6), 10, 15(6), *19*, 30, 31, 34(11), 36(11), 37, 38, 39(11), 43(11), 70(11), 83(11), 84, 89(11), 91(11), *92*, 93, 95, 102, 104, 106, 107, 108, *122*, 137, *150*
Kazarnovskaya, B. E., 207, 211(140), 213(140), *220*, *223*
Kelvin, W. T. (Lord), 124, 140
Khomovskaya, E. D., 171, *220*
Khristianovich, S. A., 155, 156, 157, 166, 172(29), 176(21), *220*
Kirchhoff, G., 140
Kolmogoroff, A. N., 2, 5, 6, 9, 10, 11, 14, *19*
Korchebokov, N. A., *220*
Kozeny, J., 183, *220*
Kozlov, V. S., 158, 162, *220*
Kuerti, G., **21–92**
Kufarev, P. P., 211, 212, *220*, *224*
Kuznetsov, V. N., *220*

L

Lagrange, J. L., 135
Lamé, J., 136, 141, 143
Laue, M. von, 125
Lavrentiev, M., 192, 196, *220*
Lees, Lester, 23
Legendre, A. M., 131

Leibenson, L. S., 202, 211, *220*
Lewis, J. A., 28(28), 84, 89(28), *92*
Lin, C. C., 1–19, 1, 2(7), 4, 6(9), 7, 9(7), 10, 11, 13(7), 15(7), 18(7), *19*
Löwner, Charles, 132, 133, 145, *149*
Loitsiansky, L. G., 6, 7, 8, 9, *19*
Lorenz, H., 95, 102, 104, 105, *122*
Love, A. E. H., 136(31), 137, 145(31), *150*
Lyon, H., 54

M

McRoberts, T. M., 197
Mangler, W., 89, 90, *92*
Massotti, A., 126, *148*
Mathews, G. B., 197
Mazkevich, D. R., 153
Meissner, E., 99
Meksyn, D., 63, 67, 69, *92*
Meleshchenko, N. T., 162, *220*
Merlin, E., 125, 132, *148*
Meyer, Th., 130, 131, 132, 134, *149*
Miche, R., 145, *151*
Michell, J. H., 136
Mikhlin, S. G., 172(29), *218*
Millionshchikov, M. D., 8, *19*, 196, 207, 215, *221*
Mises, R. von, 28(3), 31, 37, 39, 47, 90, *91*, 135(30b), 136, 141, 142, 143, 147, *150*
Mohr, O., 141
Munk, M. M., 133, *149*
Muskat, M., 207, 210, 215, 217, *221*
Myatiev, A. N., 199, 200, *221*

N

Navier, C. L. M. H., 137
Nedriga, V. P., *221*
Nelson-Skornyakov, F. B., 165, 168, 171, 173, 175, 180, 185, 187, *221*
Neményi, P. F., **123–151**, 133, 134, 137, 138(39, 43), 139(43), 146, 147, *149*, *150*
Neuber, H., 136, 139, 140, 145, *150*, *151*
Nöther, F., 126, *148*
Numerov, S. N., 168, 171, 187, 191, *221*

O

Obukhoff, A., 5, 6
Onsager, L., 6, *19*
Oseen, C. W., 127, 129, *149*

P

Pavlov, A. T., *222*
Pavlovsky, N. N., 157, 158, 162, 163, 192, *222*
Piercy, N. A. V., 85
Platrier, C., 136
Pogrebissky, U. B., 196, *220*, *222*
Pohlhausen, E., 29(5), 37, 48, 51, 54, 55, 56, 57, 64, 65(5), 66, 78, 80, 81, 84, 88(5), *91*
Poisson, S. D., 100, 135, 138
Polubarinova-Kochina, P. Ya., **153–225**, 158, 176, 178, 180, 181, 195, 196, 197, 199, 200, 202, 203, 205, 207, 211(59), 213(59), 217(126), *220*, *222*, *225*
Poritsky, H., 133, 134, *149*
Prager, W., 141(48), 145, 146, *151*
Prandtl, L., 26, 35, 38, 48, 62, 65, 69, 73, 74, 75, 78, 130, 131, 132, 134, 141, 206
Preston, J. H., 85
Prim, R. C., 124, 125, 133, 134, 146, 147, *148*, *149*, *150*
Pucher, A., 138, *150*
Pykhachev, G. V., 202, 206, *222*, *225*

R

Rankine, W. J. M., 124, 145, *148*
Razin, K. I., *220*
Reissner, E., **93–122**, 96(8), 97(8), *122*
Reissner, H., 99
Ringleb, F., 132, *149*
Risenkampf, B. K., 166, 168, 170, 178, 195, 196, 206, *223*
Rosenblatt, A., 126, 127, *148*
Rosenhead, L., *218*
Rossbach, H., 162, *225*
Rubesin, M. W., 37, 38, 40, 42(19), 43(19), 50, 82(27), 83, 86, 87(19), 90, *92*

S

St. Venant, A. J. C. Barré de, 136, 141
Schlichting, H., 51
Schmidt, E., 142, 143, 145, 147
Schuh, H., 84, 85, 86(29), 91, *92*
Seewald, F., 137, *150*
Segal, B. I., 162, 208, 209, 210, *223*
Shchelkachev, V. N., 202, 215, 217, *223*

Shekhtman, Yu. M., *223*
Shipenko, N. I., 162, *225*
Skan, S. W., 67, 88, 90
Slichter, Ch., 171, *223*
Sokolovsky, V. V., 145, *151*
Stalder, J. R., 90
Stewartson, K., 91
Suray, Saffet, 141, *151*
Szymanski, G., 127

T

Tait, P. G., 140
Taylor, G. I., 128, 129, 131, *149*
Tedone, O., 136(31), 145(31), *150*
Temple, G. F. J., 132, *149*
Tendeland, T., 90
Tetervin, M., 28(4), *91*
Thomson, W., 136, 137
Tifford, A. N., 86, 88, *92*
Timoshenko, S., 95, *122*
Timpe, A., 136(31), 145(31), *150*
Tollmien, W., 131(24), 132, 134, 146, *149*, *151*
Topolyansky, D. B., 196, *224*
Townsend, A. A., 8, 15, *19*
Trefftz, E., 183, *219*
Trkal, V., 129, *149*
Truesdell, C. A., 148
Tsien, H. S., 30(11), 31, 34(11), 36(11), 37, 38, 39(11), 43(11), 70(11), 83(11), 84, 89(11), 91(11), *92*
Tueda, M., 96, 100, *122*

U

Uginchus, A. A., 180, *224*

V

Van Iterson, 145, *151*
Van Tuyl, Andrew, 143, 147, 148
Vedernikov, V. V., 164, 165, 166, 168, 171, 173, 175, 183, 185, *224*
Verigin, N. N., 159(185), *224*
Viguier, G., 24
Vinogradov, Yu. P., 211, 212, *220*, *224*
Voigt, W., 136
Volterra, V., 140
Voshchinin, A., 186, *224*

W

Wasserman, R. H., 133
Watson, G. N., 110(11), *122*
Weber, C., 95
Wegner, U., 146
Weinel, E., 145
von Weizsäcker, C. F., 6, *19*
Wendt, H., 32, 37, 42, 45, 70, 82, 83(13), 84, *91*
Wieghardt, K., 127, *149*
Wien, W., 30(6), *91*
Witoszynski, C., 127

Y

Young, A. D., 54, 82, 83, 91(25), *92*

Z

Zamarin, E. A., 162, 171, *225*
Zarevich, 207
Zauerbreĭ, I. I., *225*
Zhukovsky, N. E., 155, 162, 163, 164, 165, 166, 171, 174, 185, 201, *225*

Subject Index

A

Accelerated flow (boundary layer), 57f.
Asymptotic integration (boundary layer), 63
Asymptotic solution (tube bending)
  circular case, 109ff.
  elliptical case, 118ff.
Axisymmetric boundary layer flow, 89f.

B

Beltrami flows, 125f.
  generalized, 134
Berker's method (inverse methods), 128
Blasius constant, 68
Blasius function, 29, 72
Boundary layer, equations of, 28f.
Boundary layer theory, mathematics of, 37ff.

C

Canonical equations (Munk and Prim), 133
Castigliano's principle applied to bending of tubes, 95, 104, 108
Compressibility factor (Howarth), 48
Constant pressure gradient, 61
Constant velocity flows (inverse methods), 125
Contraction of oil bearing region, 211ff.
Correlation tensor, 2f.
Correlation theory, 2ff.
Crocco's integral (boundary layer), 30
Curvilinear reservoir boundary, 183ff.
Curvilinear upstream slope, 185f.

D

Dam, triangular, 172f.
Dam drained at base, 168ff.
Darcy's law, 154ff.
Depression of soil water level, 213ff.
Displacement of liquid surface (filtration), 215f.
Drain channel, trapezoidal, 173ff.

E

Elastic regime (filtration), 217
Elastostatics, inverse methods in, 136ff.
Energy equation, 25
Energy principle applied to bending of tubes, 95, 104
Equilibrium temperature, 81

F

Falkner-Skan equation, 88
Filtration,
  from channels, 183
  under apron, 161f.
Flattening effect (tubes), 94
Flow under dams (variable pressure), 210f.
Fourier integral in filtration, 195ff.
Functional analysis in filtration, 191f.

G

Gas flows, inverse methods in, 130ff.
  nonsteady, 135f.
General hodograph (filtration), 175f.

H

Heat removal (boundary layer), 34
Heat transfer characteristics (boundary layer), 32ff.
Heat transfer coefficient, 36, 71
Hilbert's problem (filtration), 187ff.
Howarth equation (turbulence), 4

I

Impermeable walls (filtration), 157
Incompressible inviscid fluids, inverse methods in, 124ff.

Incompressible viscous fluid, inverse methods in, 126ff.
Inflow to drain channel, 183
Inverse methods (filtration), 182ff.
Inverse methods, significance of, 145ff.
Isotropic turbulence, 1ff.

J

Jeffery-Hamel problem, 127

K

Kármán's approximation (tube bending), 106f.
Kármán equation (turbulence), 2
Kolmogoroff's theorem (turbulence), 5

L

Limiting line, 131
Linear differential equation theory, applied to filtration, 176ff., 197ff.
Lin's theory (turbulence), 10
Loitsiansky's theorem (turbulence), 6

M

Maximum stress (tube bending), 116
Mises transformation, 37ff.
Momentum thickness, equation of, 52
Motion of two liquids (filtration), 216
Multilayer soils, 193ff.

N

Newton's law of heat transfer, 82
Nonhorizontal impermeable rock, 186f.
Nonvanishing pressure gradients (boundary layer), 88ff.
Nusselt number, 36, 82

O

Oblique cutoff (filtration), 159f.
Oil, filtration of, 201ff.

P

Plasticity, inverse methods in, 141
Pohlhausen's calculation of heat transfer, 64, 80
Pohlhausen's method, 51
  generalization of, 48, 55ff.
Prandtl-Meyer flow, 130, 131
Prandtl's number, 28

R

Recovery factor, 66
Reservoir boundaries (filtration), 157f.
Restricted problem (boundary layer), 43, 63ff.
  for $\omega = 1$, 72ff.
  for $\omega \neq 1$, 75ff.
Reynolds' analogy, 35, 71, 75
Rigidity factor (tube bending), 113
Ringleb's solution, 132

S

Seepage line, 171ff.
Seepage on dam slope, 180ff.
Separation point (boundary layer), 56
Series expansions (boundary layer), 57ff., 60ff.
Series solution (tube bending), 102ff.
Similarity in spectral theory (turbulence), 6ff.
Simple problem (boundary layer), 29ff., 33, 70ff.
Simple wave, 130
Singularities,
  method of, in filtration, 193ff.
  method of (inverse methods), 124
Soil water motion not obeying Darcy's law, 155ff.
  without a free surface, 157ff.
  with a free surface, 163ff.
Spectral tensor, 3
Spectral theory, 2ff.
Stagnation temperature, 32
Stress fields with singularities, 137ff.
Stress function,
  Crocco, 44, 73
  Hantzsche and Wendt, 45, 70
Stress functions (inverse methods), 136f.
Sutherland's formula, 40f., 78

T

Taylor-Bateman flow, 131
Taylor flow, 128f.
Temperature recovery, 66

Three dimensional problems in filtration, 206ff.
Three dimensional sink (filtration), 196f.
Three ranges theory (turbulence), 11ff.
Total heat, 30
Trapezoidal channel (filtration), 166ff.
Tube bending, equations for, 99ff.
  circular case, 102
  elliptical case, 117
Tube problem, formulation of, 97ff.

U

Unsteady flows (filtration), 210ff.

V

Variational methods in filtration, 192f.
Viscosity laws in boundary layer theory, 40f.
Viscous compressible fluid, equations of, 23ff.
Volterra dislocation, 140

W

Wakes and jets (boundary layer), 89
Wall stress estimates, 83
Wall temperature prescribed (boundary layer), 86ff.
Well, hydraulic theory of, 199ff.
Well in heterogeneous medium, 205f.

Z

Zero order functions (Cope and Hartree), 62
Zhukovsky's problem, 163f.